Some results on backward equations and stochastic partial differential equations with singularities

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Quelques résultats sur les équations rétrogrades et équations aux dérivées partielles stochastiques avec singularités.

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Résumé

Cette thèse est consacrée à l'étude de quelques problèmes dans le domaine des équations différentielles stochastiques rétrogrades (EDSR), et leurs applications aux équations aux dérivées partielles (EDP) et à la finance.

Les deux premiers chapitres sont dédiés aux EDSR avec condition terminale singulière. Ce type d'équations a été introduit par Popier dans [83], et pose le problème de l'existence de solutions d'EDSR lorsque la condition terminale peut prendre la valeur $+\infty$ sur un ensemble de mesure non négligeable. Dans le premier chapitre, nous introduisons la notion d'équation différentielle doublement stochastique rétrograde (EDDSR) avec condition terminale singulière. Un premier travail consistera à étudier les EDDSR avec générateur monotone. Nous obtenons ensuite un résultat d'existence par un schéma d'approximation en considérant une troncature de la condition terminale. On peut vérifier assez aisément que le processus limite construit ainsi satisfait la dynamique qui nous intéresse, et possède les bonnes propriétés d'intégration sur des intervalles du type $[0, T-\delta]$. En revanche la continuité en T est plus problématique, et est obtenue par deux méthodes différentes selon les valeurs de q. La dernière partie de ce chapitre vise à établir le lien avec les équations aux dérivées partielles stochastiques (EDPS), en utilisant l'approche de type solution faible développée par Bally, Matoussi in [6].

Le second chapitre est dédié aux EDSR avec condition terminale singulière et sauts. Comme dans le chapitre précédent la partie délicate sera de prouver la continuité en T. Nous formulons des conditions suffisantes sur les sauts de la diffusion afin d'obtenir cette dernière. Une section est ensuite vouée à établir le lien entre solution minimale de l'EDSR et équations intégro-différentielles.

Enfin un dernier chapitre est consacré aux équations différentielles stochastiques rétrogrades du second ordre (2EDSR) doublement réfléchies. Nous avons cherché à établir l'existence et l'unicité de telles équations en suivant une approche identique à celle développée dans [89] et [90] par Soner, Touzi et Zhang. Pour ce faire, il nous a fallu dans un premier temps nous concentrer sur le problème de réflexion par barrière supérieure des 2EDSR. Ce travail se situe dans la continuité du papier [66], où Matoussi, Possamaï et Zhou considèrent le problème de la réflexion par barrière inférieure. Contrairement aux EDSR classiques, le caractère non linéaire propre aux 2EDSR rend le problème de réflexion non symétrique. Nous traitons ce problème essentiellement sous l'hypothèse que la barrière supérieure est une semi-martingale dont le processus à variation finie admet une décomposition de Jordan. Nous avons ensuite combiné ces résultats à ceux développés dans [66] afin de donner une définition et un cadre aux 2EDSR doublement réfléchies. L'unicité est établie comme conséquence directe d'une propriété de représentation. L'existence est obtenue en utilisant les espaces shiftés, et les distributions de probabilité conditionnelles régulières. Enfin une partie application aux jeux de Dynkin et options Israéliennes sous incertitude est traitée dans la dernière section.

Mots-clés: équations différentielles stochastiques rétrogrades du second ordre, analyse stochastique quasi-sûre, jeux de Dynkin sous incertitude, équations différentielles doublement stochastiques rétrogrades, condition terminale singulière, équations aux dérivées partielles stochastiques, solutions de viscosité, équations différentielles stochastiques rétrogrades avec sauts, équations intégro-différentielles.

Abstract

This thesis is devoted to the study of some problems in the field of backward stochastic differential equations (BSDE), and their applications to partial differential equations (PDE) and finance.

The two first chapters are dedicated to BSDE with singular terminal conditions. This kind of equations has been introduced by Popier in [83], and put forward the problem of existence of BSDE solutions when the terminal condition is allowed to take the value $+\infty$ on a non-negligible set. In the first chapter, we introduce the notion of backward doubly stochastic differential equations (BDSDE) with singular terminal condition. A first work consists in studying the case of BDSDE with monotone generator. We then obtain existing result by an approximating scheme built considering a truncation of the terminal condition. We can easily verify that the limit process obtained satisfy the dynamic we are interested in, and possess the good integrability properties on all intervals of type $[0, T - \delta]$. On the other hand continuity in T is more problematic and is proved with two different methods according to q values. The last part of this chapter aim to establish the link with stochastic partial differential equations (SPDE), using a weak solution approach developed by Bally, Matoussi in [6].

The second chapter is devoted to the BSDEs with singular terminal conditions and jumps. As in the previous chapter the tricky part will be to prove continuity in T. We formulate sufficient conditions on the jumps in order to obtain it. A section is then dedicated to establish a link between a minimal solution of our BSDE and partial integro-differential equations (PIDE).

A last chapter is dedicated to doubly reflected second order backward stochastic differential equations (2DRBSDE). We have been looking to establish existence and uniqueness for such equations by following a similar approach developed by Soner, Touzi and Zhang in [89], [90]. In order to obtain this, we had to focus first on the upper reflection problem for 2BSDEs. This work is the continuity of [66], where Matoussi, Possamaï and Zhou considered the lower reflection problem. Unlike classical BSDEs, the non-linearity nature of 2BSDE make this problem non symmetrical. We treat this problem essentially under the hypothesis that the upper barrier is a semi-martingale whose finite variation process admits a Jordan decomposition. We combined then these results to those obtained in [66] to give a definition and a wellposedness context to 2DRBSDE. Uniqueness is established as a straight consequence of a representation property. Existence is obtained using shifted spaces, and regular conditional probability distributions. A last part is then consecrated to the link with some Dynkin games and Israeli options under uncertainty.

Keywords: second order backward stochastic differential equations, quasi-sure stochastic analysis, Dynkin games with uncertainty, backward doubly stochastic differential equations, singular terminal conditions, stochastic partial differential equations, viscosity solutions, backward stochastic differential equations with jumps, partial-integral differential equations.

Contents

Chapter 1

Introduction

This thesis is devoted to the study of some issues in the field of backward stochastic differential equations (BSDEs).

These equations has been first introduced by Bismut in 1973 [17], and then generalized by Pardoux, Peng in 1990 [73]. A standard BSDE can be expressed as follows:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s,\, t \in [0,T],\, \mathbb{P} - a.s.$$

W is a standard Brownian Motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with $(\mathcal{F}_t)_{t\geq 0}$ the usual augmented filtration. The generator f is a progressively measurable function and ξ the terminal value is supposed to be $\mathcal{F}_T$-measurable. The main difference with a classic SDE is that the terminal value is fixed instead of the initial one. As a consequence the process Y is defined at each time t backward in time. The solution is constituted by a couple of $\mathcal{F}$-adapted processes (Y, Z), with $\mathcal{F}$ the natural filtration associated to the Brownian motion. Since at anytime Y_t in its explicit writing let appear $\mathcal{F}_T$-measurable processes, the adapted nature of Y is not obvious, Z plays that precise adaptability role.

In their seminal work [73], Pardoux and Peng established existence and uniqueness results for this BSDE, for a generator f uniformly Lipschitz in y and z and ξ, $f(.,0,0)$ square integrable. There has been since many works whose goal was to weaken those hypothesis.

The BSDEs are widely studied because they are deeply linked with the theory of partial differential equations (PDEs). Indeed in a Markovian context, one can consider the following forward-backward system:

$$\begin{aligned} Y_t &= g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s \\ X_t &= x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s \end{aligned} \tag{1.0.1}$$

The functions f and g here are assumed to be deterministic. Consider now the following PDE:

$$\begin{aligned} &\partial_t u(t,x) + \mathcal{L}u(t,x) + f(t,x,u(t,x), \nabla u(t,x)\sigma(t,x)) = 0,\, (t,x) \in [0,T) \times \mathbb{R} \\ &u(T,.) = g(.) \end{aligned} \tag{1.0.2}$$

$\mathcal{L}$, the infinitesimal generator of X is defined as follows:

$$\mathcal{L}\phi := \frac{1}{2} Tr(\sigma\sigma^* D^2\phi) + b.\nabla\phi$$

Recall ∇ and D^2 are respectively the gradient and the Hessian matrix w.r.t. x.

Then a simple application of Itô formula shows that if u is a strong solution to the PDE 1.0.2, then $(Y_t, Z_t) := (u(t,X_t), \nabla u(t,X_t)\sigma(t,X_t))$ provides a solution to the BSDE 1.0.1. This probabilistic representation of the PDE allows one, finally to use probabilistic methods to get numerical simulations of PDE solutions. Note that through this representation we get a constraint on the class of PDE associated to a classical BSDE: one can only get this link with quasi-linear PDE. These equations appear naturally in many problems in finance, for a more complete overview regarding BSDEs and their applications, we refer the reader to the books [28], [34], [63] and [76].

This thesis will be structured as follows: the two first chapters are devoted to singular terminal condition BSDEs, once in a doubly stochastic framework and then considering BSDEs with jumps. A third chapter is dedicated to the issue of doubly reflected 2BSDEs.

1.1 BSDEs with singular terminal condition, SPDEs and PIDEs

1.1.1 BDSDEs and SPDEs

In 1994, Pardoux and Peng [74] have introduced a new class of BSDE, called "doubly stochastic" (BDSDE in short), which are related to a quasilinear parabolic stochastic partial differential equations (SPDE in short). Roughly speaking, the BSDE becomes:

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r)\, dr + \int_t^T g(r, Y_r, Z_r)\, \overleftarrow{dB_r} - \int_t^T Z_r dW_r, \ 0 \le t \le T, \tag{1.1.3}$$

where B is a Brownian motion, independent of W, and $\overleftarrow{dB_r}$ stands for the backward Itô integral (see [75], [57]). These BSDEs are connected with the following type of stochastic PDEs: for $(t,x) \in [0,T] \times \mathbb{R}^d$

$$\begin{aligned} u(t,x) &= h(x) + \int_t^T \left[\mathcal{L}u(s,x) + f(s,x,u(s,x),(\nabla u\sigma)(s,x))\right] ds \\ &+ \int_t^T g(s,x,u(s,x),(\nabla u\sigma)(s,x))\overleftarrow{dB_r}. \end{aligned} \tag{1.1.4}$$

In [74], considering a square integrable terminal condition ξ the authors established existence and uniqueness results for a BDSDE under the following Lipschitz hypothesis:

$$|f(t,y_1,z_1) - f(t,y_2,z_2)|^2 \le C(|y_1-y_2|^2 + |z_1-z_2|^2) \tag{Lip}$$

$$|g(t,y_1,z_1) - g(t,y_2,z_2)|^2 \le C|y_1-y_2|^2 + \varepsilon|z_1-z_2|^2, \quad 0 < \varepsilon < 1 \tag{Lip2}$$

and under smoothness assumptions of the coefficients, Pardoux and Peng proved existence and uniqueness of a classical solution for SPDE (1.1.4) and established the connection with solutions of BDSDE (1.1.3). Indeed if we consider a Markovian framework, with a terminal condition such as $\xi = h(X_T)$ with $h \in C^3(\mathbb{R}^d)$, growing at most like a polynomial function at infinity and X a forward diffusion as in (1.0.1), with coefficients satisfying: $b, \sigma \in C^3$ and have bounded derivatives of all order. Then $u(t,x) := Y_t^{t,x}$ is the unique solution of SPDE (1.1.4).

In [6], Bally and Matoussi prove existence and uniqueness for solutions of SPDE (1.1.4) under weaker assumptions: f and g are supposed to be Lipschitz continuous functions. Moreover they manage to find a weak formulation for SPDE and establish the link with BDSDE. To summarize, they prove that the SPDE (1.1.4) has a unique solution u in a Sobolev space and that $Y_t = u(t, X_t)$, $Z_t = (\sigma^*\nabla u)(t, X_t)$ is the unique solution of BDSDE (1.1.3). One of the key tools is the theory of stochastic flows already used by Kunita [55, 56]. Under other smooth assumptions on g, Buckdahn and Ma [21, 22] developed the notion of stochastic viscosity solution for SPDE and proved existence and uniqueness of a (stochastically) bounded viscosity solution.

1.1.2 BSDEs with jumps and PIDEs

Jumps in BSDEs were first introduced by Li and Tang in [93] considering a Lipschitz generator and a fixed point approach as in [73]. A solution of a standard BSDE with jumps is a triple (Y, Z, U) satisfying:

$$Y_t = \xi - \int_t^T f(s, Y_s, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e)\tilde{\mu}(ds, de) \tag{1.1.5}$$

The additional stochastic integral considered, is an integral with respect to a compensated Poisson measure $\tilde{\mu}$. Existence and uniqueness results has been established for a Lipschitz generator. Since then, this branch has been deepened by many authors (for instance see [28],[52], [86]), these kind of BSDEs are linked to a specific class of PDE, partial integro differential equations (PIDEs) as studied in [9].

Barles, Buckdahn, Pardoux, indeed did exhibit the link between a BSDE with jumps in a Markovian setup with a forward process X satisfying the following dynamic:

$$dX_s = b(X_s)ds + \sigma(X_s)dW_s + \int_E \beta(X_{s^-}, e)\tilde{\mu}(ds, de)$$

and the following PIDE:

$$\begin{aligned}
&-\partial_t u(t,x) - Ku(t,x) - f(t,x,u(t,x), \nabla u\sigma(t,x), Bu(t,x)) = 0, \ (t,x) \in [0,T) \times \mathbb{R}^d \\
&u(T,x) = g(x), \ x \in \mathbb{R}^d
\end{aligned}$$

with K and B defined by:

$$K\phi(x) := \mathcal{L}\phi(x) + \int_E (\phi(x + \beta(x,e)) - \phi(x) - \nabla\phi(x).\beta(x,e))\lambda(de)$$

$$B\phi(x) := \int_E (\phi(x + \beta(x,e)) - \phi(x))\gamma(x,e)\lambda(de)$$

It is important to note that in order to obtain this result the authors get a very specific dependance in the jumps for their generator explicitly defined by the form of operator B.

1.1.3 BSDEs with singular terminal condition

The notion of backward stochastic differential equations (BSDE) with singular terminal condition was first introduced in [83] in 2006. Indeed , Popier in [83] studied the behavior of solutions of BSDE when the terminal condition is allowed to take infinite values on non-negligible set i.e.

$$\mathbb{P}(\xi = +\infty \text{ or } \xi = -\infty) > 0.$$

The idea behind the notion of solutions for such equations is the following, we want the couple (Y, Z) to satisfy the classical dynamic, and to belong to the right spaces but strictly before T, and additionally we impose continuity in T for Y.
This has been done in order to obtain a generalization of the Feynman-Kac formula for non linear partial differential equations (PDE) of the following form:

$$\partial_t u + \mathcal{L}u - u|u|^q = 0 \tag{1.1.6}$$

with $\mathcal{L}$ defined as above.

This PDE has been widely studied by PDE arguments (see among others Baras and Pierre [7] and Marcus and Veron in [64]). It is shown in [64] that every solution of this PDE can be characterized by a final trace which is a couple $(\mathcal{S}, \mu)$ where $\mathcal{S}$ is a closed subset of $\mathbb{R}^d$ and μ a non-negative Radom measure on $\mathbb{R}^d \backslash \mathcal{S}$.
The final trace can also be represented by a positive, outer regular Borel measure ν, and ν is not necessary locally bounded. The two representations are related by:

$$\forall A \subset \mathbb{R}^m, A \text{ Borel}, \begin{cases} \nu(A) = \infty & \text{if } A \cap \mathcal{S} \neq \emptyset \\ \nu(A) = \mu(A) & \text{if } A \subset \mathcal{R}. \end{cases}$$

The set $\mathcal{S}$ is the set of singular final points of u and it corresponds to a "blow-up" set of u.
Dynkin and Kuznetsov [30] and Le Gall [59] have proved same kind of results but in a probabilistic framework by using the superprocesses theory.

In [83], the generator f is given by: $f(y) = -y|y|^q$. A minimal solution is constructed by an non decreasing approximation scheme. The main difficulty is the proof of continuity of the minimal solution Y at time T. In general Y is a supersolution and the converse property was proved under stronger sufficient conditions. See also [3] and [40] for other examples and for the link with a stochastic control problem with terminal constraint. More recently aswell, Kruse and Popier in [53] exposed an interesting application to optimal position targeting linked with portfolio liquidation issue. In these papers the continuity problem is not studied.

Let us expose the example developed in [3]. Consider a functional J as follows:

$$J(x) = \mathbb{E}\left[\int_0^T (\eta_t|\dot{x}_t|^p + \gamma_t|x_t|^p)dt\right]$$

and want to minimize it over all absolutely continuous paths $(x_t)_{t\in[0,T]}$ starting in x_0 and ending at 0 at T. This kind of control problems occurs for instance when an economic agent has to close a position of x_0 assets shares in a market with a stochastic price impact.
The first term $\int_0^T (\eta_t|\dot{x}_t|^p)dt$ can be interpreted as the liquidity cost induced by closing the position, where η is a stochastic price impact factor. The second term is then seen as a measure of the risk associated to the open position.

It has been shown in [3] that if η and γ satisfy some integrability condition, the following BSDE:

$$dY_t = \left((p-1)\frac{Y_t^q}{\eta_t^{q-1}} - \gamma_t\right)dt + Z_t dW_t$$

where q is the conjugate of p, with the singular terminal condition:

$$\lim_{t \to T} Y_t = \infty$$

has a minimal solution. Moreover Y provides an optimal control given by:

$$x_t^* = x_0 e^{-\int_0^t (\frac{Y_s}{\eta_s})^{q-1} ds}$$

and the value function is given by: $J(x^*) = |x_0| Y_0$.

1.1.4 Motivation, formulation of the problem

In the chapter 2 our aim is twofold, first we prove existence and uniqueness of the solution of a BDSDE with monotone generator f, then we extend the results of [83] to BDSDE in order to obtain a solution for a SPDE with singular terminal condition h. To our best knowlegde the closest result concerning our first issue is in Aman [2]. Nevertheless we think that there is a lack in this paper (proof 4.2). Indeed for monotone BSDE ($g = 0$) the existence of a solution relies on the solvability of the BSDE:

$$Y_t = \xi + \int_t^T f(r, Y_r)\, dr - \int_t^T Z_r dW_r, \ 0 \leq t \leq T.$$

See among other the proof of Theorem 2.2 and Proposition 2.4 in [72]. To obtain a solution for this BSDE, the main trick is to truncate the coefficients with suitable truncation functions in order to have a bounded solution Y (see Proposition 2.2 in [18]). This can not be done for a general BDSDE. Indeed take for example ($\xi = f = 0$ and $g = 1$):

$$Y_t = \int_t^T \overleftarrow{dB_r} - \int_t^T Z_r dW_r = B_T - B_t, \ 0 \leq t \leq T,$$

with $Z = 0$. Thus in order to prove existence of a solution for (1.1.3), one can not directly follow the scheme of [72]. The first part of this chapter is devoted to the existence of a solution for a monotone BDSDE (see Section 2.2) in the space $\mathcal{E}^2$ (see definition in the next section). To realize this project we will restrict the class of functions f: they should satisfy a polynomial growth condition (as in [18]). Until now we do not know how to extend this to general growth condition as in [72] or [19].
The second goal of this work is to extend the results of [83] to the doubly stochastic framework. We will consider the generator $f(y) = -y|y|^q$ with $q \in \mathbb{R}_+^*$ and a real $\mathcal{F}_T^W$-measurable and non negative random variable ξ such that:

$$\mathbb{P}(\xi = +\infty) > 0. \tag{1.1.7}$$

And we want to find a solution to the following BDSDE:

$$Y_t = \xi - \int_t^T Y_s |Y_s|^q ds + \int_t^T g(s, Y_s, Z_s) \overleftarrow{dB_s} - \int_t^T Z_s dW_s. \tag{1.1.8}$$

The scheme to construct a solution is almost the same as in [83]. Let us emphasize one of main technical difficulties. If $g = 0$, we can use the conditional expectation w.r.t. $\mathcal{F}_t$ to withdraw the martingale part. If $g \neq 0$, this trick is useless and we have to be very careful when we want almost sure property of the solution. This BDSDE is connected with the stochastic PDE with terminal condition h: for any $0 \leq t \leq T$

$$\begin{aligned} u(t,x) &= h(x) + \int_t^T (\mathcal{L}u(s,x) - u(s,x)|u(s,x)|^q)\, ds \\ &\quad + \int_t^T g(s, u(s,x), \sigma(s,x)\nabla u(s,x)) \overleftarrow{dB_s}. \end{aligned} \tag{1.1.9}$$

If h is a smooth function we could use the result of Pardoux and Peng [74]. But here we will assume that $\mathcal{S} = \{h = +\infty\}$ is a closed non empty set and thus we will precise the notion of solution for (1.1.9) in this case. Roughly speaking we will show that there is a minimal solution u in the sense that u belongs to a Sobolev space and is a weak solution of the SPDE on any interval $[0, T-\delta]$, $\delta > 0$ and satisfies the terminal condition: $u(t,x)$ goes to $h(x)$ also in a weak sense as t goes to T.

In Chapter 3 we would like to provide existence and unicity results for the following BSDE:

$$Y_t = \xi - \int_t^T Y_s|Y_s|^q ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e)\tilde{\mu}(ds,de) \tag{1.1.10}$$

where the terminal condition ξ satisfies

$$\mathbb{P}(\xi = +\infty) > 0. \tag{1.1.11}$$

It is already established that such a BSDE has a unique solution when the terminal condition ξ belongs to $L^p(\Omega, \mathcal{F}_T, \mathbb{P})$, $p > 1$ (see among others [9], [28], [52] or [93]). We would like to extend this result when the terminal condition is singular, i.e. verifies (1.1.11). More precisely in [53] (see Theorem 17 below) it is proved that the BSDE (1.1.10) with singular terminal condition has a minimal super-solution (Y, Z, U) such that

$$\liminf_{t \to T} Y_t \geq \xi.$$

Since a priori estimates are not hard to establish, to obtain our existence results, we essentially want to find sufficient conditions to have a.s.

$$\lim_{t \to T} Y_t = \xi. \tag{1.1.12}$$

This problem was studied in [83] when there is no jump. In the Markovian framework and for $q > 2$, Equality (1.1.12) was proved. Here we will follow the same idea. But we need other technical assumptions on the jumps of the solution X of the forward SDE and the set $\mathcal{S} = \{\xi = +\infty\}$.
The second part is devoted to the study of the related partial integro differential equation (PIDE in short): for any $x \in \mathbb{R}^d$, $u(T,x) = g(x)$ and for any $(t,x) \in [0,T[\times \mathbb{R}^d$

$$\frac{\partial}{\partial t} u(t,x) + \mathcal{L}u(t,x) + \mathcal{I}(t,x,u) - u(t,x)|u(t,x)|^q = 0 \tag{1.1.13}$$

where $\mathcal{L}$ is given above and $\mathcal{I}$ is a integro-differential operator:

$$\mathcal{I}(t,x,\phi) = \int_E [\phi(x + h(t,x,e)) - \phi(x) - (\nabla \phi)(x) h(t,x,e)] \lambda(de).$$

If there is no jump, in [83] the following link is established between the solution $Y^{t,x}$ of the BSDE (1.1.10) and the viscosity solution u of the PIDE (1.1.13):

$$u(t,x) = Y_t^{t,x}$$

This relation is also obtained in [9] in the case where there are jumps and if the terminal function g is of linear growth. Moreover several papers have studied the existence and the uniqueness of the solution of such PIDE (see among others [1], [10], [13] or [48]).
To our best knowledges the study of (1.1.13) in the jump case and with a singularity at time T is completely new. Singularity means that the set $\{g = +\infty\}$ is non empty. There is no probabilistic representation of such PIDE using superprocesses. Hence the second aim of the chapter is to prove that this minimal solution Y is the probabilistic representation of the minimal positive viscosity solution u of the PIDE. Moreover we will show that the sufficient conditions for (1.1.12) are also sufficient to ensure "continuity at time T" of u.

1.1.5 Contributions to BDSDEs

Let us now precise our notations. W and B are independent Brownian motions defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}^k$ and $\mathbb{R}^m$. Let $\mathcal{N}$ denote the class of $\mathbb{P}$-null sets of $\mathcal{F}$. For each $t \in [0,T]$, we define

$$\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B$$

where for any process η, $\mathcal{F}_{s,t}^\eta = \sigma\{\eta_r - \eta_s;\ s \leq r \leq t\} \vee \mathcal{N}$, $\mathcal{F}_t^\eta = \mathcal{F}_{0,t}^\eta$. As in [74] we define the following filtration $(\mathcal{G}_t,\ t \in [0,T])$ by:

$$\mathcal{G}_t = \mathcal{F}_t^W \vee \mathcal{F}_{0,T}^B.$$

ξ is a $\mathcal{F}_T^W$-measurable and $\mathbb{R}^d$-valued random variable.
We define by $\mathbb{H}^p(0,T;\mathbb{R}^n)$ the set of (classes of $d\mathbb{P} \times dt$ a.e. equal) n dimensional jointly measurable random processes $(X_t,\ t \geq 0)$ which satisty:

1. $\mathbb{E}\left(\int_0^T |X_t|^2 dt\right)^{p/2} < +\infty$
2. X_t is $\mathcal{G}_t$-measurable for a.e. $t \in [0,T]$.

We denote similarly by $\mathbb{S}^p(0,T;\mathbb{R}^n)$ the set of continuous n dimensional random processes which satisfy:

1. $\mathbb{E}\left(\sup_{t\in[0,T]} |X_t|^p\right) < +\infty$
2. X_t is $\mathcal{G}_t$-measurable for any $t \in [0,T]$.

$\mathbb{B}^p(0,T)$ is the product $\mathbb{S}^p(0,T;\mathbb{R}^d) \times \mathbb{H}^p(0,T;\mathbb{R}^{d\times k})$. $(Y,Z) \in \mathcal{E}^p(0,T)$ if $(Y,Z) \in \mathbb{B}^p(0,T)$ and Y_t and Z_t are $\mathcal{F}_t$-measurable. Finally $C^{p,q}([0,T] \times \mathbb{R}^d;\mathbb{R})$ denotes the space of $\mathbb{R}$-valued functions defined on $[0,T] \times \mathbb{R}^d$ which are p-times continuously differentiable in $t \in [0,T]$ and q-times continuously differentiable in $x \in \mathbb{R}^d$. $C_b^{p,q}([0,T] \times \mathbb{R}^d;\mathbb{R})$ is the subspace of $C^{p,q}([0,T] \times \mathbb{R}^d;\mathbb{R})$ in which all functions have uniformly bounded partial derivatives; and $C_c^{p,q}([0,T] \times \mathbb{R}^d;\mathbb{R})$ the subspace of $C^{p,q}([0,T] \times \mathbb{R}^d;\mathbb{R})$ in which the functions have a compact support w.r.t. $x \in \mathbb{R}^d$.

Now we precise our assumptions on f and g. The functions f and g are defined on $[0,T] \times \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times k}$ with values respectively in $\mathbb{R}^d$ and $\mathbb{R}^{d\times m}$. Moreover we consider the following assumptions.

Assumptions (A).

- The function $y \mapsto f(t,y,z)$ is continuous and there exists a constant μ such that for any (t,y,y',z) a.s.
$$\langle y-y', f(t,y,z)-f(t,y',z)\rangle \leq \mu|y-y'|^2. \tag{A1}$$
- There exists K_f such that for any (t,y,z,z') a.s.
$$|f(t,y,z)-f(t,y,z')|^2 \leq K_f|z-z'|^2. \tag{A2}$$
- There exists $C_f \geq 0$ and $p>1$ such that
$$|f(t,y,z)-f(t,0,z)| \leq C_f(1+|y|^p). \tag{A3}$$
- There exists a constant $K_g \geq 0$ and $0<\varepsilon<1$ such that for any (t,y,y',z,z') a.s.
$$|g(t,y,z)-g(t,y',z')|^2 \leq K_g|y-y'|^2+\varepsilon|z-z'|^2. \tag{A4}$$
- Finally for any (t,y,z), $f(t,y,z)$ and $g(t,y,z)$ are $\mathcal{F}_t$-measurable with
$$\mathbb{E}\int_0^T \left(|f(t,0,0)|^2+|g(t,0,0)|^2\right) dt < +\infty. \tag{A5}$$

Remember that from [74] if f also satisfies: there exists $\tilde{K}_f$ such that for any (t,y,y',z) a.s.
$$|f(t,y,z)-f(t,y',z)| \leq \tilde{K}_f|y-y'| \tag{1.1.14}$$
then there exists a unique solution $(Y,Z) \in \mathcal{E}^2(0,T)$ to the BDSDE (1.1.3). Note that (1.1.14) implies that
$$|f(t,y,z)-f(t,0,z)| \leq \tilde{K}_f|y|,$$
thus the growth assumption (A3) on f is satisfied with $p=1$.
We emphasize the fact that the assumptions (A2), (A4) are exactly (Lip),(Lip2) from [74] thus are somewhat standard when dealing with the doubly stochastic framework.
In Section 2.2 we will prove the following result.

Theorem 1. *Under Assumptions **(A)** and if the terminal condition ξ satisfies*
$$\mathbb{E}(|\xi|^2) < +\infty, \tag{1.1.15}$$
the BDSDE (1.1.3) has a unique solution $(Y,Z) \in \mathcal{E}^2(0,T)$.

Using the paper of Aman [2], this result can be extended to the L^p case: for $p \in (1,2)$
$$\mathbb{E}\int_0^T \left(|f(t,0,0)|^p+|g(t,0,0)|^p\right) dt + \mathbb{E}(|\xi|^p) < +\infty.$$

There exists a unique solution in $\mathcal{E}^p(0,T)$.
The next sections are devoted to the singular case. The generator f will be supposed to be deterministic and given by: $f(y) = -y|y|^q$ for some $q > 0$. The aim is to prove existence of a solution for BDSDE (1.1.8) when the non negative random variable ξ satisfies (1.1.7). A possible extension of the notion of solution for a BDSDE with singular terminal condition could be the following (see Definition 1 in [83]).

Definition 1 (Solution of the BDSDE (1.1.8))**.** *Let $q > 0$ and ξ a $\mathcal{F}_T^W$-measurable non negative random variable satisfying condition* (1.1.7)*. We say that the process (Y, Z) is a solution of the BDSDE* (1.1.8) *if (Y, Z) is such that (Y_t, Z_t) is $\mathcal{F}_t$-measurable and:*

(D1) for all $0 \leq s \leq t < T$*:* $Y_s = Y_t - \int_s^t Y_r|Y_r|^q dr + \int_s^t g(r, Y_r, Z_r)\overleftarrow{dB_r} - \int_s^t Z_r dW_r$*;*

(D2) for all $t \in [0, T[$*,* $\mathbb{E}\left(\sup_{0\leq s\leq t} |Y_s|^2 + \int_0^t \|Z_r\|^2 dr\right) < +\infty$*;*

(D3) $\mathbb{P}$*-a.s.* $\lim_{t\to T} Y_t = \xi$*.*

A solution is said non negative if a.s. for any $t \in [0, T]$, $Y_t \geq 0$.

In order to define a wellposedness theory for DBSDE with singular terminal condition, we follow the idea initiated in [83], but we'll have to deal with the backward integral term which intervene in a non-trivial way in some situations.

To obtain an a priori estimate of the solution we will assume that $g(t, y, 0) = 0$ for any (t, y) a.s. This condition will ensure that our solutions will be non negative and bounded on any time interval $[0, T - \delta]$ with $\delta > 0$. Without this hypothesis, integrability of the solution would be more challenging. In Section 2.3, we will prove the following result.

Theorem 2. *There exists a process (Y, Z) satisfying Conditions (D1) and (D2) of Definition 8 and such that Y has a limit at time T with*

$$\lim_{t\to T} Y_t \geq \xi.$$

Moreover this solution is minimal: if $(\widetilde{Y}, \widetilde{Z})$ is a non negative solution of (1.1.8)*, then a.s. for any t, $\widetilde{Y}_t \geq Y_t$.*

It means in particular that Y_t has a left limit at time T.
In general we are not able to prove that (D3) holds. As in [83], we give sufficient conditions for continuity and we prove it in the Markovian framework. Hence the first hypothesis on ξ is the following:

$$\xi = h(X_T), \tag{H1}$$

where h is a function defined on $\mathbb{R}^d$ with values in $\overline{\mathbb{R}^+}$ such that the set of singularity $\mathcal{S} = \{h = +\infty\}$ is closed; and where X_T is the value at $t = T$ of a diffusion process or more precisely the solution of a stochastic differential equation (in short SDE):

$$X_t = x + \int_0^t b(r, X_r)dr + \int_0^t \sigma(r, X_r)dW_r, \text{ for } t \in [0, T]. \tag{1.1.16}$$

We will always assume that b and σ are defined on $[0,T]\times\mathbb{R}^d$, with values respectively in $\mathbb{R}^d$ and $\mathbb{R}^{d\times k}$, are measurable w.r.t. the Borelian σ-algebras, and that there exists a constant $K>0$ s.t. for all $t\in[0,T]$ and for all $(x,y)\in\mathbb{R}^d\times\mathbb{R}^d$:

1. Lipschitz condition:

$$|b(t,x)-b(t,y)|+\|\sigma(t,x)-\sigma(t,y)\|\leq K|x-y|; \tag{L}$$

2. Growth condition:

$$|b(t,x)|\leq K(1+|x|) \text{ and } \|\sigma(t,x)\|\leq K(1+|x|). \tag{G}$$

It is well known that under the previous assumptions, Equation (1.1.16) has a unique strong solution X.
We denote $\mathcal{R}=\mathbb{R}^d\setminus\mathcal{S}$. The second hypothesis on ξ is: for all compact set $\mathcal{K}\subset\mathcal{R}=\mathbb{R}^d\setminus\{h=+\infty\}$

$$h(X_T)\mathbf{1}_{\mathcal{K}}(X_T)\in\ L^1\left(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R}\right). \tag{H2}$$

Unfortunately the above assumptions are not sufficient to prove continuity if $q\leq 2$. Thus we add the following conditions in order to use Malliavin calculus and to prove Equality (2.4.41).

1. The functions σ and b are bounded: there exists a constant K s.t.

$$\forall(t,x)\in[0,T]\times\mathbb{R}^d,\quad |b(t,x)|+\|\sigma(t,x)\|\leq K. \tag{B}$$

2. The second derivatives of $\sigma\sigma^*$ belongs to L^∞:

$$\frac{\partial^2\sigma\sigma^*}{\partial x_i\partial x_j}\in L^\infty([0,T]\times\mathbb{R}^d). \tag{D}$$

3. $\sigma\sigma^*$ is uniformly elliptic, i.e. there exists $\lambda>0$ s.t. for all $(t,x)\in[0,T]\times\mathbb{R}^d$:

$$\forall y\in\mathbb{R}^d,\ \sigma\sigma^*(t,x)y.y\geq\lambda|y|^2. \tag{E}$$

4. h is continuous from $\mathbb{R}^d$ to $\overline{\mathbb{R}_+}$ and:

$$\forall M\geq 0,\ h \text{ is a Lipschitz function on the set } \mathcal{O}_M=\{|h|\leq M\}. \tag{H3}$$

Theorem 3. *Under Assumptions* (H1), (H2) *and* (L) *and if*

- *either* $q>2$ *and* (G)*;*
- *or* (B), (D), (E) *and* (H3)*;*

the minimal non negative solution (Y,Z) *of* (1.1.8) *satisfies (D3): a.s.*

$$\lim_{t\to T}Y_t=\xi.$$

Note that these conditions are identical to [83], nevertheless none of the proofs of this result could be adapted directly to our doubly stochastic context. Moreover it seems surprising to us that we do not get additional hypothesis on g by introducing a doubly stochastic framework, other than (Lip), (Lip2) from [74].
Finally in section 2.5, we show that this minimal solution (Y, Z) of (1.1.8) is connected to the minimal weak solution u of the SPDE (1.1.9). More precisely $X^{t,x}$ is the solution of the SDE (1.1.16) with initial condition x at time t and $(Y^{t,x}, Z^{t,x})$ is the minimal solution of the BDSDE (1.1.8) with singular terminal condition $\xi = h(X_T^{t,x})$.
Let us define the space $\mathcal{H}(0,T)$ as in [6]. We take the following weight function $\rho : \mathbb{R}^d \to \mathbb{R}$: for $\kappa > d$

$$\rho(x) = \frac{1}{(1+|x|)^\kappa}. \tag{1.1.17}$$

The constant κ will be fixed later. $\mathcal{H}(0,T)$ is the set of the random fields $\{u(t,x);\ 0 \leq t \leq T,\ x \in \mathbb{R}^d\}$ such that $u(t,x)$ is $\mathcal{F}^B_{t,T}$-measurable for each (t,x), u and $\sigma^*\nabla u$ belong to $L^2((0,T)\times\Omega\times\mathbb{R}^d;\ ds\otimes d\mathbb{P}\otimes\rho(x)dx)$. On $\mathcal{H}(0,T)$ we consider the following norm

$$\|u\|_2^2 = \mathbb{E}\int_0^T\int_{\mathbb{R}^d}\left(|u(s,x)|^2 + |(\sigma^*\nabla u)(s,x)|^2\right)\rho(x)dx.$$

Theorem 4. *The random field u defined by $u(t,x) = Y_t^{t,x}$ belongs to $\mathcal{H}(0,T-\delta)$ for any $\delta > 0$ and is a weak solution of the SPDE (1.1.9) on $[0,T-\delta]\times\mathbb{R}^d$. At time T, u satisfies a.s. $\liminf_{t\to T} u(t,x) \geq h(x)$.*
Moreover under the same assumptions of Theorem 3, for any function $\phi \in C_c^\infty(\mathbb{R}^d)$ with support included in $\mathcal{R}$, then

$$\lim_{t\to T}\mathbb{E}\left(\int_{\mathbb{R}^d} u(t,x)\phi(x)dx\right) = \int_{\mathbb{R}^d} h(x)\phi(x)dx.$$

Finally u is the minimal non negative solution of (1.1.9).

In order to prove this result, we proceeded in a few steps. First we noticed that (1.1.9) with terminal function $h \wedge n$ has an unique weak solution u^n thanks to [6] and this solution coincide with the solution of the forward-backward system in the following sense: $Y_s^{n,t,x} = u^n(s, X_s^{t,x})$, $Z_s^{n,t,x} = (\sigma^*\nabla u^n)(s, X_s^{t,x})$. We define then u as the limit of the random fields u^n. Then it is easy to deduce an upper bound for u on $[0, T-\delta]$, and by dominated convergence we get that u satisfies the two first properties of a weak solution. And since we have good a priori estimates we can obtain that u satisfies the last property by a monotone convergence.
The last point: $\lim_{t\to T}\mathbb{E}\left(\int_{\mathbb{R}^d} u(t,x)\phi(x)dx\right) = \int_{\mathbb{R}^d} h(x)\phi(x)dx$ is obtained by proving inequalities in both ways and standard arguments.

1.1.6 Contributions to BSDEs with jumps

Our setting is the same as in [9]. In the following we will consider $W = (W_t)_{t\in\mathbb{R}_+}$ the standard Brownian Motion on $\mathbb{R}^k$, μ a Poisson random measure on $\mathbb{R}_+ \times E$ with compensator $dt\lambda(de)$. Here $E := \mathbb{R}^\ell\setminus\{0\}$, $\mathcal{E}$ its Borel field and we assume that we have a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$ and the filtration $(\mathcal{F}_t,\ t \geq 0)$ is generated by the two independent processes W and μ and we assume that $\mathcal{F}_0$ contains all $\mathbb{P}$-null elements of $\mathcal{F}$. We will denote $\tilde{\mu}$ is the compensated measure: for any $A \in \mathcal{E}$ such that $\lambda(A) < +\infty$, then $\tilde{\mu}([0,t]\times A) = \mu([0,t]\times A) - t\lambda(A)$ is a martingale. λ is assumed to be a σ-finite measure on $(E,\mathcal{E})$ satisfying

$$\int_E (1\wedge |e|^2)\lambda(de) < +\infty.$$

In this chapter for a given $T\geq 0$, we denote:

- $\mathcal{P}$: the predictable σ-field on $\Omega\times[0,T]$ and

$$\widetilde{\mathcal{P}} = \mathcal{P}\otimes\mathcal{E}$$

- On $\widetilde{\Omega} = \Omega\times[0,T]\times E$, a function that is $\widetilde{\mathcal{P}}$-measurable, is called predictable. $G_{loc}(\mu)$ is the set of $\widetilde{\mathcal{P}}$-measurable functions ψ on $\widetilde{\Omega}$ such that for any $t\geq 0$ a.s.

$$\int_0^t\int_E (|\psi_s(e)|^2\wedge|\psi_s(e)|)\lambda(de) < +\infty.$$

- $\mathcal{D}$ (resp. $\mathcal{D}(0,T)$): the set of all progressively measurable càdlàg processes on $\mathbb{R}_+$ (resp. on $[0,T]$).

- $\mathbb{S}^p(0,T)$ is the space of all processes $X\in\mathcal{D}(0,T)$ such that

$$\mathbb{E}\left(\sup_{t\in[0,T]}|X_t|^p\right) < +\infty.$$

  For simplicity, $X_* = \sup_{t\in[0,T]}|X_t|$.

- $\mathbb{H}^p(0,T)$ is the subspace of all processes $X\in\mathcal{D}(0,T)$ such that

$$\mathbb{E}\left[\left(\int_0^T |X_t|^2 dt\right)^{p/2}\right] < +\infty.$$

- $\mathbb{L}^p_\mu(0,T) = L^p_\mu(\Omega\times(0,T)\times E)$: the set of processes $\psi\in G_{loc}(\mu)$ such that

$$\mathbb{E}\left[\left(\int_0^T\int_E |\psi_s(e)|^2\lambda(de)ds\right)^{p/2}\right] < +\infty.$$

- $L^p_\lambda = L^p(E,\lambda;\mathbb{R}^d)$: the set of measurable functions $\psi: E\to\mathbb{R}^d$ such that

$$\|\psi\|^p_{L^p_\lambda} = \int_E |\psi(e)|^p\lambda(de) < +\infty.$$

- $\mathbb{B}^p_\mu(0,T) = \mathbb{S}^p(0,T)\times\mathbb{H}^p(0,T)\times\mathbb{L}^p_\mu(0,T)$.

Definition 2. *Let $q>0$ and ξ a $\mathcal{F}_t$-measurable non negative random variable such that $\mathbb{P}(\xi=\infty)>0$. We say (Y,Z,U) is a solution of (1.1.10) with singular terminal condition ξ if*

- *(Y,Z,U) belongs to $\mathbb{B}^2(0,t)$ for any $t<T$.*

- *for all* $0 \leq s \leq t < T$*:*

$$Y_s = Y_t - \int_s^t Y_r |Y_r|^q dr - \int_s^t Z_r dW_r - \int_s^t \int_E U_s(e) \tilde{\mu}(ds, de).$$

- Y *is continuous at* T*: a.s.*

$$\lim_{t \to T} Y_t = \xi.$$

If we consider the same approximating scheme as in [53], we define Y the limit process of Y^n where Y^n satisfies the BSDE with jumps associated to the data $(\xi \wedge n, U)$. Then we can obtain this proposition.

Proposition 1. *The process* Y *can be written as follows:*

$$Y_t = \left(q(T-t) + \mathbb{E}^{\mathcal{F}_t} \left[\frac{1}{\xi^q} \right] - \phi_t \right)^{-1/q}$$

where ϕ *is a non negative supermartingale. As a consequence, the process* Y *has a left limit in* T*.*

We suppose now our terminal condition is of the form: $\xi = g(X_T)$. The function g is defined on $\mathbb{R}^d$ with values in $\mathbb{R}_+ \cup \{+\infty\}$ and we denote

$$\mathcal{S} := \{x \in \mathbb{R}^d \quad s.t. \quad g(x) = \infty\}$$

the set of singularity points for the terminal condition induced by g. This set $\mathcal{S}$ is supposed to be closed. We also denoted by Γ the boundary of $\mathcal{S}$. The process X is the solution of a SDE with jumps:

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s + \int_0^t \int_E h(s, X_{s_-}, e) \tilde{\mu}(de, ds). \tag{1.1.18}$$

The coefficients $b : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d \times k}$ and $h : \Omega \times [0,T] \times \mathbb{R}^d \times E \to \mathbb{R}^d$ satisfy

Assumptions (B):

1. b, σ and h are jointly continuous w.r.t. (t,x) and Lipschitz continuous w.r.t. x uniformly in t, e or ω, i.e. there exists a constant C such that for any $(\omega, t, e) \in \Omega \times [0,T] \times E$, for any x and y in $\mathbb{R}^d$: a.s.

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq C|x-y| \tag{L}$$

and

$$\int_E |h(t,x,e) - h(t,y,e)|^2 \lambda(de) \leq C|x-y|. \tag{B2}$$

2. b, σ and h growth at most linearly:

$$|b(t,x)| + |\sigma(t,x)| \leq C(1+|x|). \tag{G}$$

3. h is bounded w.r.t. t and x and there exists a constant C_h such that a.s.

$$|h(t,x,e)| \leq C_h(1 \wedge |e|). \tag{B4}$$

Now the second hypothesis on ξ is: for all compact set $\mathcal{K} \subset \mathbb{R}^d \setminus \mathcal{S}$

$$g(X_T)\mathbf{1}_{\mathcal{K}}(X_T) \in \; L^1(\Omega, \mathcal{F}_T, \mathbb{P}). \tag{1.1.19}$$

Assumption (C):

- The boundary $\partial\mathcal{S} = \Gamma$ is compact and of class C^2.
- For any $x \in \mathcal{S}$, any $s \in [0,T]$ and λ-a.s.

  $$x + h(s,x,e) \in \mathcal{S}.$$

  Furthermore there exists a constant $\kappa > 0$ such that if $x \in \Gamma$, then for any $s \in [0,T]$, $d(x+h(s,x,e),\Gamma) \geq \kappa$, λ-a.s.

Theorem 5. *Under Assumptions **(B)** and **(C)**, Y is continuous at T.*

To have more regularity on u we add some conditions on the coefficients.

1. σ and b are bounded: there exists a constant C s.t.

$$\forall (t,x) \in [0,T] \times \mathbb{R}^d, \quad |b(t,x)| + \|\sigma(t,x)\| \leq C; \tag{B}$$

2. $\sigma\sigma^*$ is uniformly elliptic, i.e. there exists $\lambda > 0$ s.t. for all $(t,x) \in [0,T] \times \mathbb{R}^d$:

$$\forall y \in \mathbb{R}^d,\; \sigma\sigma^*(t,x)y.y \geq \lambda |y|^2. \tag{E}$$

Definition 3 (Viscosity solution with singular data)**.** *A function u is a viscosity solution of* (1.1.13) *with terminal data g if u is a viscosity solution on $[0,T[\times\mathbb{R}^d$ and satisfies:*

$$\lim_{(t,x)\to(T,x_0)} u(t,x) = g(x_0). \tag{1.1.20}$$

Theorem 6. *The function u is the minimal viscosity solution of* (1.1.13) *with terminal data g.*

1.1.7 Perspectives

In the singular terminal condition framework, we assume the generator to be known and explicit as:

$$f(\omega,t,y,z) := -y|y|^q$$

This allows us when the terminal condition is deterministic to obtain the following solution to our BSDE (which becomes an ODE):

$$y_t^C = (q(T-t) + \frac{1}{C})^{-\frac{1}{q}},\; Z \equiv 0$$

Then by comparison theorem we easily get majorations for our processes Y^n, $Y_t^n \leq y_t^n$. And the choice of our generator allows us to obtain some nice properties on y^n that is: $y^n \leq y^\infty$, $\forall n \in \mathbb{N}$ and $y_t^\infty < \infty$, $\forall t < T$.

One question that raises obviously is: can we describe a class of generators f for which existence of a solution stands for a BSDE with singular terminal condition?
Noting that the previous ODE with a general generator f depending only on y can be written:

$$\begin{aligned} & y_t = C + \int_t^T f(y_s)ds \\ \Leftrightarrow \quad & y'(t) = -f(y_t) \\ \Leftrightarrow \quad & \int_t^T \frac{y'(s)}{f(y_s)}ds = -(T-t) \\ \Leftrightarrow \quad & \int_{y(t)}^\infty -\frac{1}{f(u)}du = t - T < \infty \end{aligned}$$

We conjecture that our results can be extended to continuous and monotonic, functions f such that $1/f \in L^1([0;\infty))$. Nevertheless we face some difficulties, regarding a priori estimates, existence of a limit and continuity at T our previous arguments does not work anymore.
Some recent works from [53] have made one step in that direction, indeed the authors have weakened the conditions on the generator f, they consider a more general constraint type:

$$f(t, y, \psi) \leq -\frac{\rho - 1}{a_t}|y|^r + f(t, 0, \psi)$$

and a future work could be to study the continuity at time T of a solution under this hypothesis. Another generalization that could be made would be to remove the sign assumption on ξ. In the two problems of singular terminal condition we assume $\xi \geq 0$. We do think this is not necessary and probably there is way to adapt the part 5 of [83] in order to obtain the same result with a slightly more general terminal condition.

1.2 2DBSDEs and related game options

1.2.1 RBSDEs, DRBSDEs and applications

The notion of Reflected BSDE (RBSDE) has been introduced by El Karoui, Kapoudjian, Pardoux, Peng, Quenez in [33]. In addition with the parameters (f, ξ), we consider a barrier process S. In that situation the solution is a triple (Y, Z, K), with K non-decreasing, such that:

$$\begin{aligned} & Y_t = \xi + \int_t^T f_s(Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dB_s,\ t \in [0, T],\ \mathbb{P} - a.s. \\ & Y_t \geq S_t,\ t \in [0, T],\ \mathbb{P} - a.s. \\ & \int_0^T (Y_t - S_t)dK_t = 0,\ \mathbb{P} - a.s. \end{aligned}$$

The role of the process K here is to push upward Y in order to keep it above the barrier S. The last condition is known as the Skorohod condition and guarantees that the process K acts in a minimal way, that is to say only when the process Y reaches the lower barrier S. The RBSDEs are linked with the problem of pricing American contingent claim by replication, especially in constrained markets (see [35] and [32] for more details).

Building upon these results, Cvitanić and Karatzas [27] have introduced the notion of BSDEs with two reflecting barriers. Roughly speaking, in [27] (see also [32] and [46] to name but a few) the authors have looked for a solution to a BSDE whose Y component is forced to stay between two prescribed processes L and U, $(L \leq U)$. More precisely, they were looking for a quadruple of progressively measurable processes (Y, Z, K^+, K^-), where K^+ and K^- are in addition non-decreasing such that

$$\begin{aligned}
&Y_t = \xi + \int_t^T f_s(Y_s, Z_s)ds + K_T^- - K_t^- - K_T^+ + K_t^+ - \int_t^T Z_s dB_s,\ t \in [0,T],\ \mathbb{P}-a.s.\\
&L_t \leq Y_t \leq U_t,\ t \in [0,T],\ \mathbb{P}-a.s.\\
&\int_0^T (Y_t - L_t)dK_t^- = \int_0^T (U_t - Y_t)dK_t^+ = 0,\ \mathbb{P}-a.s.
\end{aligned}$$

These BSDEs have been developed especially in connection with Dynkin games, mixed differential games and *recallable* options (see ([32], [45], [44] and [41]). It is now established that under quite general assumptions, including in models with jumps, existence of a solution to a (simply) reflected BSDE is guaranteed under mild conditions, whereas existence of a solution to a *doubly reflected* BSDE (DRBSDE for short) is equivalent to the so-called *Mokobodski condition*. This condition essentially postulates the existence of a quasimartingale between the barriers (see in particular [43]). Note that in [43] authors did demonstrate that existence of local solutions is equivalent to a weakened Mokobodski condition, roughly speaking the existence of a difference of non-negative supermartingales between the barriers. As for uniqueness of solutions, it is guaranteed under mild integrability conditions (see e.g. [43, Remark 4.1]). However, for practical purposes, existence and uniqueness are not the only relevant results and may not be enough. For instance, one can consider the problem of pricing convertible bonds in finance using the DRBSDE theory (see [15], [16] and [26]). In this case, the state-process (first component) Y may be interpreted in terms of an arbitrage price process for the bond. As demonstrated in [16], the mere existence of a solution to the related DRBSDE is a result with important theoretical consequences in terms of pricing and hedging the bond. Yet, in order to give further developments to these results in Markovian set-ups, Crépey and Matoussi [26] have established *bound and error estimates* and *comparison theorem* for DRBSDE, which require more regularity assumptions on the barriers.

1.2.2 2BSDEs

More recently, motivated by numerical methods for fully nonlinear PDEs, second order BSDEs (2BSDEs for short) were introduced by Cheredito, Soner, Touzi and Victoir in [24]. Then Soner, Touzi and Zhang [89] proposed a new formulation and obtained a complete theory of existence and uniqueness for such BSDEs. The main novelty in their approach is that they require that the solution verifies the equation $\mathbb{P}-a.s.$ for every probability measure $\mathbb{P}$ in a non-dominated set. Their approach therefore shares many connections with the deep theory of quasi-sure analysis initiated by Denis and Martini [29] and the G-expectations developed by Peng [78].

Intuitively speaking (we refer the reader to [89] for more details), the solution to a 2BSDE with generator F and terminal condition ξ can be understood as a supremum in some sense of the classical BSDEs with the same generator and terminal condition, but written under the different probability measures considered. Following this intuition, a non-decreasing process K is added to the solution and it somehow pushes (in a minimal way) the solution so that it stays above the solutions of the classical BSDEs.

Following these results and motivated by the pricing of American contingent claims in markets with volatility uncertainty, Matoussi, Possamaï and Zhou [66] used the methodology of [89] to introduce a notion of reflected second order BSDEs, and proved existence and uniqueness in the case of a lower obstacle. The fact that they consider only lower obstacles was absolutely crucial. Indeed, as mentioned above, in that case, the effects due to the reflection and the second order act in the same direction, in the sense that they both force the solution to stay above some processes. One therefore only needs to add a non-decreasing process to the solution of the equation. However, as soon as one tries to consider upper obstacles, the two effects start to counterbalance each other and the situation changes drastically. This case was thus left open in [66]. On a related note, we would like to refer the reader to the very recent article [31], which gives some specific results for the optimal stopping problem under a non-linear expectation (which roughly corresponds to a 2RBSDE with generator equal to 0). However, since it is a "sup-sup" problem, it is only related to the lower reflected 2BSDEs. Even more recently and after the completion of this work, Nutz and Zhang [68] managed to treat the same problem of optimal stopping under non-linear expectations but now with an "inf-sup" formulation, which, as shown by Proposition 2, is related to upper reflected 2BSDEs.

1.2.3 Motivation

The first aim of this chapter is to extend the results of [66] to the case of doubly reflected second-order BSDEs when we assume enough regularity on one of the barriers (as in [26]) and that the two barriers are completely separated (as in [42] and [43]). In that case, we show that the right way to define a solution is to consider a 2BSDE where we add a process V which has only bounded variations (see Definition 6). Our next step towards a theory of existence and uniqueness is then to understand as much as possible how and when this bounded variation process acts. Our key result is obtained in Proposition 4.3.5, and allows us to obtain a special Jordan decomposition for V, in the sense that we can decompose it into the difference of two non-decreasing processes which never act at the same time. Thanks to this result, we are then able to obtain a priori estimates and a uniqueness result. Next, we reuse the methodology of [66] to construct a solution.

We also show that these objects are related to non-standard optimal stopping games, thus generalizing the connection between DRBSDEs and Dynkin games first proved by Cvitanić and Karatzas [27]. Finally, we show that the second order DRBSDEs allow to obtain super and sub-hedging prices for American game options (also called Israeli options) in financial markets with volatility uncertainty and that, under a technical assumption, they provide solutions of what we call uncertain Dynkin games.

1.2.4 Main results

Let $\Omega := \{\omega \in C([0,T],\mathbb{R}^d) : \omega_0 = 0\}$ be the canonical space equipped with the uniform norm $\|\omega\|_\infty := \sup_{0\le t\le T}|\omega_t|$, B the canonical process, $\mathbb{P}_0$ the Wiener measure, $\mathbb{F} := \{\mathcal{F}_t\}_{0\le t\le T}$ the filtration generated by B, and $\mathbb{F}^+ := \{\mathcal{F}_t^+\}_{0\le t\le T}$ the right limit of $\mathbb{F}$. A probability measure $\mathbb{P}$ will be called a local martingale measure if the canonical process B is a local martingale under $\mathbb{P}$. Then, using results of Bichteler [14] (see also Karandikar [50] for a modern account), the quadratic variation $\langle B\rangle$ and its density $\widehat{a}$ can be defined pathwise, and such that they coincide with the usual definitions under any local martingale measure.

With the intuition of modeling, volatility uncertainty, we let $\overline{\mathcal{P}}_W$ denote the set of all local martingale measures $\mathbb{P}$ such that

$$\langle B\rangle \text{ is absolutely continuous in } t \text{ and } \widehat{a} \text{ takes values in } \mathbb{S}_d^{>0},\ \mathbb{P}-a.s., \tag{1.2.21}$$

where $\mathbb{S}_d^{>0}$ denotes the space of all $d\times d$ real valued positive definite matrices.

However, since this set is too large for our purpose (in particular there are examples of measures in $\overline{\mathcal{P}}_W$ which do not satisfy the martingale representation property, see [88] for more details), we will concentrate on the following subclass $\mathcal{P}_S$ consisting of

$$\mathbb{P}^\alpha := \mathbb{P}_0\circ(X^\alpha)^{-1} \text{ where } X_t^\alpha := \int_0^t \alpha_s^{1/2}dB_s,\ t\in[0,T],\ \mathbb{P}_0-a.s., \tag{1.2.22}$$

for some $\mathbb{F}$-progressively measurable process α taking values in $\mathbb{S}_d^{>0}$ with $\int_0^T|\alpha_t|dt < +\infty$, $\mathbb{P}_0-a.s.$ This subset has the convenient property that all its elements do satisfy the martingale representation property and the Blumenthal $0-1$ law (see [88] for details) which are crucial tools for the BSDE theory.

We consider a map $H_t(\omega,y,z,\gamma) : [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times D_H\to\mathbb{R}$, where $D_H\subset\mathbb{R}^{d\times d}$ is a given subset containing 0, whose Fenchel transform w.r.t. γ is denoted by

$$F_t(\omega,y,z,a) := \sup_{\gamma\in D_H}\left\{\frac{1}{2}\mathrm{Tr}(a\gamma) - H_t(\omega,y,z,\gamma)\right\} \text{ for } a\in\mathbb{S}_d^{>0},$$
$$\widehat{F}_t(y,z) := F_t(y,z,\widehat{a}_t) \text{ and } \widehat{F}_t^0 := \widehat{F}_t(0,0).$$

We denote by $D_{F_t(y,z)} := \{a,\ F_t(\omega,y,z,a) < +\infty\}$ the domain of F in a for a fixed (t,ω,y,z). As in [89] we fix a constant $\kappa\in(1,2]$ and restrict the probability measures in $\mathcal{P}_H^\kappa\subset\overline{\mathcal{P}}_S$

Definition 4. *$\mathcal{P}_H^\kappa$ consists of all $\mathbb{P}\in\overline{\mathcal{P}}_S$ such that*

$$\underline{a}^{\mathbb{P}} \le \widehat{a} \le \bar{a}^{\mathbb{P}},\ dt\times d\mathbb{P}-a.s. \text{ for some } \underline{a}^{\mathbb{P}},\bar{a}^{\mathbb{P}}\in\mathbb{S}_d^{>0},\ \text{and } \phi_H^{2,\kappa} < +\infty,$$

where

$$\phi_H^{2,\kappa} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}_{0\le t\le T}^{\mathbb{P}}\left(\mathbb{E}_t^{H,\mathbb{P}}\left[\int_0^T|\hat{F}_s^0|^\kappa ds\right]\right)^{\frac{2}{\kappa}}\right].$$

Definition 5. *We say that a property holds $\mathcal{P}_H^\kappa$-quasi-surely ($\mathcal{P}_H^\kappa$-q.s. for short) if it holds $\mathbb{P}$-a.s. for all $\mathbb{P}\in\mathcal{P}_H^\kappa$.*

We now state the main assumptions on the function F which will be our main interest in the sequel

Assumption (D):

(i) The domain $D_{F_t(y,z)} = D_{F_t}$ is independent of (ω, y, z).

(ii) For fixed (y, z, a), F is $\mathbb{F}$-progressively measurable in D_{F_t}.

(iii) We have the following uniform Lipschitz-type property in y and z

$$\forall (y, y', z, z', t, a, \omega), \ \left|F_t(\omega, y, z, a) - F_t(\omega, y', z', a)\right| \leq C\left(\left|y - y'\right| + \left|a^{1/2}\left(z - z'\right)\right|\right).$$

(iv) F is uniformly continuous in ω for the $||\cdot||_\infty$ norm.

(v) $\mathcal{P}_H^\kappa$ is not empty.

The following spaces and the corresponding norms will be used throughout the chapter. With the exception of the space $\mathbb{L}_H^{p,\kappa}$, they all are immediate extensions of the usual spaces to the quasi-sure setting.

For $p \geq 1$, $L_H^{p,\kappa}$ denotes the space of all $\mathcal{F}_T$-measurable scalar r.v. ξ with

$$\|\xi\|_{L_H^{p,\kappa}}^p := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P}\left[|\xi|^p\right] < +\infty.$$

$\mathbb{H}_H^{p,\kappa}$ denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}^d$-valued processes Z with

$$\|Z\|_{\mathbb{H}_H^{p,\kappa}}^p := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P}\left[\left(\int_0^T |\widehat{a}_t^{1/2} Z_t|^2 dt\right)^{\frac{p}{2}}\right] < +\infty.$$

$\mathbb{D}_H^{p,\kappa}$ denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}$-valued processes Y with

$$\mathcal{P}_H^\kappa - q.s. \text{ càdlàg paths, and } \|Y\|_{\mathbb{D}_H^{p,\kappa}}^p := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P}\left[\sup_{0\leq t\leq T} |Y_t|^p\right] < +\infty,$$

where càdlàg is the french acronym for "right-continuous with left-limits".

$\mathbb{I}_H^{p,\kappa}$ denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}$-valued processes K null at 0 with

$$\mathcal{P}_H^\kappa - q.s., \text{ càdlàg and non-decreasing paths, and } \|K\|_{\mathbb{I}_H^{p,\kappa}}^p := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P}\left[(K_T)^p\right] < +\infty.$$

$\mathbb{V}_H^{p,\kappa}$ denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}$-valued processes V null at 0 with paths which are $\mathcal{P}_H^\kappa - q.s.$ càdlàg and of bounded variation, and such that

$$\|V\|_{\mathbb{V}_H^{p,\kappa}}^p := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P}\left[(\mathrm{Var_{0,T}(V)})^p\right] < +\infty.$$

For each $\xi \in L_H^{1,\kappa}$, $\mathbb{P}\in\mathcal{P}_H^\kappa$ and $t\in[0,T]$ denote

$$\mathbb{E}_t^{H,\mathbb{P}}[\xi] := \underset{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})}{\operatorname{ess\,sup}^\mathbb{P}} \ \mathbb{E}_t^{\mathbb{P}'}[\xi] \text{ where } \mathcal{P}_H^\kappa(t^+,\mathbb{P}) := \left\{\mathbb{P}' \in \mathcal{P}_H^\kappa : \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_t^+\right\}.$$

Here $\mathbb{E}_t^{\mathbb{P}}[\xi] := E^{\mathbb{P}}[\xi|\mathcal{F}_t]$. Then we define for each $p \geq \kappa$,

$$\mathbb{L}_H^{p,\kappa} := \left\{\xi \in L_H^{p,\kappa} : \|\xi\|_{\mathbb{L}_H^{p,\kappa}} < +\infty\right\} \text{ where } \|\xi\|_{\mathbb{L}_H^{p,\kappa}}^p := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\underset{0\leq t\leq T}{\text{ess sup}}^{\mathbb{P}} \left(\mathbb{E}_t^{H,\mathbb{P}}[|\xi|^\kappa]\right)^{\frac{p}{\kappa}}\right].$$

We denote by $\mathrm{UC}_b(\Omega)$ the collection of all bounded and uniformly continuous maps $\xi : \Omega \to \mathbb{R}$ with respect to the $\|\cdot\|_\infty$-norm, and we let

$$\mathcal{L}_H^{p,\kappa} := \text{the closure of } \mathrm{UC}_b(\Omega) \text{ under the norm } \|\cdot\|_{\mathbb{L}_H^{p,\kappa}}, \text{ for every } 1 \leq \kappa \leq p.$$

Finally, for every $\mathbb{P} \in \mathcal{P}_H^\kappa$, and for any $p \geq 1$, $L^p(\mathbb{P})$, $\mathbb{H}^p(\mathbb{P})$, $\mathbb{D}^p(\mathbb{P})$, $\mathbb{I}^p(\mathbb{P})$ and $\mathbb{V}^p(\mathbb{P})$ will denote the corresponding usual spaces when there is only one measure $\mathbb{P}$.

First, we consider a process S which will play the role of the upper obstacle. We will always assume that S verifies the following properties

Assumption (E):

(i) S is $\mathbb{F}$-progressively measurable and càdlàg.

(ii) S is uniformly continuous in ω in the sense that for all t

$$|S_t(\omega) - S_t(\widetilde{\omega})| \leq \rho\left(\|\omega - \widetilde{\omega}\|_t\right), \ \forall\ (\omega, \widetilde{\omega}) \in \Omega^2,$$

for some modulus of continuity ρ and where we define $\|\omega\|_t := \sup_{0\leq s\leq t} |\omega(s)|$.

(iii) S is a semimartingale for every $\mathbb{P} \in \mathcal{P}_H^\kappa$, with the decomposition

$$S_t = S_0 + \int_0^t P_s dB_s + A_t^{\mathbb{P}},\ \mathbb{P} - a.s.,\ \text{for all } \mathbb{P} \in \mathcal{P}_H^\kappa, \tag{1.2.23}$$

where the $A^{\mathbb{P}}$ are bounded variation processes with Jordan decomposition $A^{\mathbb{P},+} - A^{\mathbb{P},-}$ and

$$\zeta_H^{2,\kappa} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \left(\mathbb{E}^{\mathbb{P}}\left[\underset{0\leq t\leq T}{\text{ess sup}}^{\mathbb{P}}\ \mathbb{E}_t^{H,\mathbb{P}}\left[\left(\int_t^T \left|\widehat{a}_s^{1/2} P_s\right|^2 ds\right)^{\kappa/2} + \left(A_T^{\mathbb{P},+}\right)^\kappa\right]\right]\right)^2 < +\infty.$$

(iv) S satisfies the following integrability condition

$$\psi_H^{2,\kappa} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\underset{0\leq t\leq T}{\text{ess sup}}^{\mathbb{P}} \left(\mathbb{E}_t^{H,\mathbb{P}}\left[\sup_{0\leq s\leq T} |S_s|^\kappa\right]\right)^{\frac{2}{\kappa}}\right] < +\infty.$$

Next, we also consider a lower obstacle L which will be assumed to verify

Assumption (F):

(i) L is a $\mathbb{F}$-progressively measurable càdlàg process.

(ii) L is uniformly continuous in ω in the sense that for all t and for some modulus of continuity ρ

$$|L_t(\omega) - L_t(\widetilde{\omega})| \leq \rho\left(\|\omega - \widetilde{\omega}\|_t\right),\ \forall\ (\omega, \widetilde{\omega}) \in \Omega^2.$$

(iii) For all $t \in [0,T]$, we have

$$L_t < S_t \text{ and } L_{t^-} < S_{t^-},\ \mathcal{P}_H^\kappa - q.s.$$

(iv) We have the following integrability condition

$$\varphi_H^{2,\kappa} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}_{0\leq t\leq T}{}^{\mathbb{P}}\left(\mathbb{E}_t^{H,\mathbb{P}}\left[\left(\sup_{0\leq s\leq T}(L_s)^+\right)^\kappa\right]\right)^{\frac{2}{\kappa}}\right] < +\infty. \tag{1.2.24}$$

We shall consider the following second order doubly reflected BSDE (2DRBSDE for short) with upper obstacle S and lower obstacle L

$$Y_t = \xi + \int_t^T \widehat{F}_s(Y_s, Z_s)ds - \int_t^T Z_s dB_s + V_T - V_t,\ 0 \leq t \leq T,\ \mathcal{P}_H^\kappa - q.s. \tag{1.2.25}$$

In order to give the definition of the 2DRBSDE, we first need to introduce the corresponding standard doubly reflected BSDEs. Hence, for any $\mathbb{P} \in \mathcal{P}_H^\kappa$, $\mathbb{F}$-stopping time τ, and $\mathcal{F}_\tau$-measurable random variable $\xi \in \mathbb{L}^2(\mathbb{P})$, let

$$(y^{\mathbb{P}}, z^{\mathbb{P}}, k^{\mathbb{P},+}, k^{\mathbb{P},-}) := (y^{\mathbb{P}}(\tau,\xi), z^{\mathbb{P}}(\tau,\xi), k^{\mathbb{P},+}(\tau,\xi), k^{\mathbb{P},-}(\tau,\xi)),$$

denote the unique solution to the following standard DRBSDE with upper obstacle S and lower obstacle L (existence and uniqueness have been proved under these assumptions in [26] among others)

$$\begin{cases} y_t^{\mathbb{P}} = \xi + \int_t^\tau \widehat{F}_s(y_s^{\mathbb{P}}, z_s^{\mathbb{P}})ds - \int_t^\tau z_s^{\mathbb{P}} dB_s + k_\tau^{\mathbb{P},-} - k_t^{\mathbb{P},-} - k_\tau^{\mathbb{P},+} + k_t^{\mathbb{P},+},\ 0 \leq t \leq \tau,\ \mathbb{P} - a.s. \\ L_t \leq y_t^{\mathbb{P}} \leq S_t,\ \mathbb{P} - a.s. \\ \int_0^t \left(y_{s^-}^{\mathbb{P}} - L_{s^-}\right) dk_s^{\mathbb{P},-} = \int_0^t \left(S_{s^-} - y_{s^-}^{\mathbb{P}}\right) dk_s^{\mathbb{P},+} = 0,\ \mathbb{P} - a.s.,\ \forall t \in [0,T]. \end{cases} \tag{1.2.26}$$

Everything is now ready for the

Definition 6. *We say $(Y, Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$ is a solution to the 2DRBSDE (1.2.25) if*

- $Y_T = \xi$, $\mathcal{P}_H^\kappa - q.s.$
- $\forall \mathbb{P} \in \mathcal{P}_H^\kappa$, *the process $V^{\mathbb{P}}$ defined below has paths of bounded variation $\mathbb{P} - a.s.$*

$$V_t^{\mathbb{P}} := Y_0 - Y_t - \int_0^t \widehat{F}_s(Y_s, Z_s)ds + \int_0^t Z_s dB_s,\ 0 \leq t \leq T,\ \mathbb{P} - a.s. \tag{1.2.27}$$

- *We have the following minimum condition for* $0 \leq t \leq T$

$$V_t^{\mathbb{P}} + k_t^{\mathbb{P},+} - k_t^{\mathbb{P},-} = \underset{\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'}\left[V_T^{\mathbb{P}'} + k_T^{\mathbb{P}',+} - k_T^{\mathbb{P}',-}\right], \ \mathbb{P}-a.s., \ \forall \mathbb{P} \in \mathcal{P}_H^{\kappa}. \quad (1.2.28)$$

- $L_t \leq Y_t \leq S_t$, $\mathcal{P}_H^{\kappa} - q.s.$

Moreover, if there exists an aggregator for the family $(V^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_H^{\kappa}}$*, that is to say a progressively measurable process* V *such that for all* $\mathbb{P} \in \mathcal{P}_H^{\kappa}$*,*

$$V_t = V_t^{\mathbb{P}}, \ t \in [0,T], \ \mathbb{P}-a.s.,$$

then we say that (Y,Z,V) *is a solution to the 2DRBSDE* (1.2.25).

We have similarly as in Theorem 4.4 of [89]

Theorem 7. *Let Assumption **(D)** hold. Assume* $\xi \in \mathbb{L}_H^{2,\kappa}$ *and that* (Y,Z) *is a solution to the* 2*DRBSDE* (1.2.25)*. Then, for any* $\mathbb{P} \in \mathcal{P}_H^{\kappa}$ *and* $0 \leq t_1 < t_2 \leq T$*,*

$$Y_{t_1} = \underset{\mathbb{P}' \in \mathcal{P}_H^{\kappa}(t_1^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}), \ \mathbb{P}-a.s. \quad (1.2.29)$$

Consequently, the 2*DRBSDE* (1.2.25) *has at most one solution in* $\mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$*.*

We are able to deduce a comparison Theorem easily from the classical one for DRBSDEs (see for instance [61]) and the representation (1.2.29) above.

Theorem 8. *Let Assumptions **(D)**, **(E)** and **(F)** hold. For* $i = 1,2$*, let* (Y^i, Z^i) *be the solutions to the 2DRBSDE* (1.2.25) *with terminal condition* ξ^i*, upper obstacle* S *and lower obstacle* L*. Then, there exists a constant* C_κ *depending only on* κ*,* T *and the Lipschitz constant of* F *such that*

$$\left\|Y^1 - Y^2\right\|_{\mathbb{D}_H^{2,\kappa}} \leq C \left\|\xi^1 - \xi^2\right\|_{\mathbb{L}_H^{2,\kappa}}$$

$$\left\|Z^1 - Z^2\right\|_{\mathbb{H}_H^{2,\kappa}}^2 + \sup_{\mathbb{P}\in\mathcal{P}_H^{\kappa}} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}\left|V_t^{\mathbb{P},+,1} - V_t^{\mathbb{P},+,2}\right|^2 + \sup_{0\leq t\leq T}\left|V_t^{\mathbb{P},-,1} - V_t^{\mathbb{P},-,2}\right|^2\right]$$
$$\leq C \left\|\xi^1 - \xi^2\right\|_{\mathbb{L}_H^{2,\kappa}}\left(\left\|\xi^1\right\|_{\mathbb{L}_H^{2,\kappa}} + \left\|\xi^1\right\|_{\mathbb{L}_H^{2,\kappa}} + (\phi_H^{2,\kappa})^{1/2} + (\psi_H^{2,\kappa})^{1/2} + (\varphi_H^{2,\kappa})^{1/2} + (\zeta_H^{2,\kappa})^{1/2}\right).$$

Now that we have proved the representation (1.2.29) and the a priori estimates of Theorems 4.3.7 and 4.3.8, we can show, as in the classical framework, that the solution Y of the 2DRBSDE is linked to some kind of Dynkin game. For any $t \in [0,T]$, denote $\mathcal{T}_{t,T}$ the set of $\mathbb{F}$-stopping times taking values in $[t,T]$.

Proposition 2. *Let* (Y,Z) *be the solution to the above* 2*DRBSDE* (1.2.25)*. For any* $(\tau,\sigma) \in \mathcal{T}_{0,T}$*, define*

$$R_\tau^\sigma := S_\tau 1_{\tau<\sigma} + L_\sigma 1_{\sigma\leq\tau,\sigma<T} + \xi 1_{\tau\wedge\sigma=T}.$$

Then for each $t \in [0,T]$*, for all* $\mathbb{P} \in \mathcal{P}_H^{\kappa}$*, we have* $\mathbb{P}-a.s.$

$$Y_t = \underset{\mathbb{P}' \in \mathcal{P}_H^{\kappa}(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \underset{\tau\in\mathcal{T}_{t,T}}{\operatorname{ess\,inf}} \underset{\sigma\in\mathcal{T}_{t,T}}{\operatorname{ess\,sup}} \mathbb{E}_t^{\mathbb{P}'}\left[\int_t^{\tau\wedge\sigma} \widehat{F}_s(y_s^{\mathbb{P}'}, z_s^{\mathbb{P}'})ds + R_\tau^\sigma\right]$$
$$= \underset{\mathbb{P}' \in \mathcal{P}_H^{\kappa}(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \underset{\sigma\in\mathcal{T}_{t,T}}{\operatorname{ess\,sup}} \underset{\tau\in\mathcal{T}_{t,T}}{\operatorname{ess\,inf}} \mathbb{E}_t^{\mathbb{P}'}\left[\int_t^{\tau\wedge\sigma} \widehat{F}_s(y_s^{\mathbb{P}'}, z_s^{\mathbb{P}'})ds + R_\tau^\sigma\right].$$

Moreover, for any $\gamma \in [0,1]$, we have $\mathbb{P}-a.s.$

$$Y_t = \underset{\tau\in\mathcal{T}_{t,T}}{\operatorname{ess\,inf}}\ \underset{\sigma\in\mathcal{T}_{t,T}}{\operatorname{ess\,sup}}\ \mathbb{E}_t^{\mathbb{P}}\left[\int_t^{\tau\wedge\sigma}\widehat{F}_s(Y_s,Z_s)ds + K_{\tau\wedge\sigma}^{\mathbb{P},\gamma} - K_t^{\mathbb{P},\gamma} + R_\tau^\sigma\right]$$
$$= \underset{\sigma\in\mathcal{T}_{t,T}}{\operatorname{ess\,sup}}\ \underset{\tau\in\mathcal{T}_{t,T}}{\operatorname{ess\,inf}}\ \mathbb{E}_t^{\mathbb{P}}\left[\int_t^{\tau\wedge\sigma}\widehat{F}_s(Y_s,Z_s)ds + K_{\tau\wedge\sigma}^{\mathbb{P},\gamma} - K_t^{\mathbb{P},\gamma} + R_\tau^\sigma\right],$$

where

$$K_t^{\mathbb{P},\gamma} := \gamma\int_0^t \mathbf{1}_{y_{s^-}^{\mathbb{P}}<S_{s^-}}dV_s^{\mathbb{P}} + (1-\gamma)\int_0^t \mathbf{1}_{Y_{s^-}>L_{s^-}}dV_s^{\mathbb{P}}.$$

Furthermore, for any $\mathbb{P}\in\mathcal{P}_H^\kappa$, the following stopping times are ε-optimal

$$\tau_t^{\varepsilon,\mathbb{P}} := \inf\left\{s\geq t,\ y_s^{\mathbb{P}}\geq S_s-\varepsilon,\ \mathbb{P}-a.s.\right\}\ \text{and}\ \sigma_t^\varepsilon := \inf\left\{s\geq t,\ Y_s\leq L_s+\varepsilon,\ \mathcal{P}_H^\kappa-q.s.\right\}.$$

Then, if we have more information on the obstacle S and its decomposition (1.2.23), we can give a more explicit representation for the processes $V^{\mathbb{P}}$, just as in the classical case (see Proposition 4.2 in [35]).

Assumption (G)

S is a semi-martingale of the form

$$S_t = S_0 + \int_0^t U_s ds + \int_0^t P_s dB_s + C_t,\ \mathcal{P}_H^\kappa - q.s.$$

where C is càdlàg process of integrable variation such that the measure dC_t is singular with respect to the Lebesgue measure dt and which admits the following decomposition $C_t = C_t^+ - C_t^-$, where C^+ and C^- are non-decreasing processes. Besides, U and V are respectively $\mathbb{R}$ and $\mathbb{R}^d$-valued $\mathcal{F}_t$ progressively measurable processes such that

$$\int_0^T (|U_t|+|P_t|^2)dt + C_T^+ + C_T^- \leq +\infty,\ \mathcal{P}_H^\kappa - q.s.$$

Theorem 9. *Let $\xi\in\mathcal{L}_H^{2,\kappa}$ and let Assumptions **(D)**, **(E)** and **(F)** hold. Then*
1) There exists a unique solution $(Y,Z)\in\mathbb{D}_H^{2,\kappa}\times\mathbb{H}_H^{2,\kappa}$ of the 2DRBSDE (1.2.25).
2) Moreover, if in addition we choose to work under either of the following model of set theory (we refer the reader to [39] for more details)

- (i) *Zermelo-Fraenkel set theory with axiom of choice (ZFC) plus the Continuum Hypothesis (CH).*
- (ii) *ZFC plus the negation of CH plus Martin's axiom.*

Then there exists a unique solution $(Y,Z,V)\in\mathbb{D}_H^{2,\kappa}\times\mathbb{H}_H^{2,\kappa}\times\mathbb{V}_H^{2,\kappa}$ of the 2DRBSDE (1.2.25).

Regarding applications, let us first recall the definition of an Israeli (or game) option, and we refer the reader to [51], [41] and the references therein for more details. An Israeli option is a contract between a broker (seller) and a trader (buyer). The specificity is that both can decide to exercise before the maturity date T. If the trader exercises first at a time t then the broker pays him the (random) amount L_t. If the broker exercises before the trader at time t, the trader will be given from him the quantity $S_t \geq L_t$, and the difference $S_t - L_t$ is as to be understood as a penalty imposed on the seller for canceling the contract. In the case where they exercise simultaneously at t, the trader payoff is L_t and if they both wait till the maturity of the contract T, the trader receives the amount ξ. In other words, this is an American option which has the specificity that the seller can also "exercise" early. This therefore is a typical Dynkin game. We assume throughout this section that the processes L and S satisfy Assumptions **(F)** and **(G)**.

To sum everything up, if we consider that the broker exercises at a stopping time $\tau \leq T$ and the trader at another time $\sigma \leq T$ then the trader receive from the broker the following payoff:

$$H(\sigma, \tau) := S_\tau 1_{\tau<\sigma} + L_\sigma 1_{\sigma\leq\tau} + \xi 1_{\sigma\wedge\tau=T}$$

Before introducing volatility uncertainty, let us first briefly recall how the fair price and the hedging of such an option is related to DRBSDEs in a classical financial market. We fix a probability measure $\mathbb{P}$, and we assume that the market contains one riskless asset, whose price is assumed w.l.o.g. to be equal to 1, and one risky asset. We furthermore assume that if the broker adopts a strategy π (which is an adapted process in $\mathbb{H}^2(\mathbb{P})$ representing the percentage of his total wealth invested in the risky asset), then his wealth process has the following expression

$$X_t^{\mathbb{P}} = \xi + \int_t^T b(s, X_s^{\mathbb{P}}, \pi_s^{\mathbb{P}})ds - \int_t^T \pi_s^{\mathbb{P}} \sigma_s dW_s \,, \mathbb{P} - a.s.$$

where W is a Brownian motion under $\mathbb{P}$, b is convex and Lipschitz with respect to (x, π). We also suppose that the process $(b(t, 0, 0))_{t\leq T}$ is square-integrable and $(\sigma_t)_{t\leq T}$ is invertible and its inverse is bounded. It was then proved in [51] and [41] that the fair price and an hedging strategy for the Israeli option described above can be obtained through the solution of a DRBSDE. More precisely, we have

Theorem 10. *The fair price of the game option and the corresponding hedging strategy are given by the pair* $(y^{\mathbb{P}}, \pi^{\mathbb{P}}) \in \mathbb{D}^2(\mathbb{P}) \times \mathbb{H}^2(\mathbb{P})$ *solving the following DRBSDE*

$$\begin{cases} y_t^{\mathbb{P}} = \xi + \int_t^T b(s, y_s^{\mathbb{P}}, \pi_s^{\mathbb{P}})ds - \int_t^T \pi_s^{\mathbb{P}} \sigma_s dW_s + k_t^{\mathbb{P}} - k_t^{\mathbb{P}}, \ \mathbb{P} - a.s. \\ L_t \leq y_t^{\mathbb{P}} \leq S_t, \ \mathbb{P} - a.s. \\ \int_0^T (y_{t^-}^{\mathbb{P}} - L_{t^-}) dk_t^{\mathbb{P},-} = \int_0^T (S_{t^-} - y_{t^-}^{\mathbb{P}}) dk_t^{\mathbb{P},+} = 0. \end{cases}$$

Moreover, for any $\varepsilon > 0$*, the following stopping times are* ε*-optimal after t for the seller and the buyer respectively*

$$D_t^{1,\varepsilon,\mathbb{P}} := \inf\left\{s \geq t, \ y_s^{\mathbb{P}} \geq S_s - \varepsilon\right\}, \ D_t^{2,\varepsilon,\mathbb{P}} := \inf\left\{s \geq t, \ y_s^{\mathbb{P}} \leq L_s + \varepsilon\right\}.$$

Definition 7. *For* $\xi \in \mathbb{L}_H^{2,\kappa}$*, we consider the following type of equations satisfied by a pair of progressively-measurable processes* (Y, Z)

- $Y_T = \xi, \ \mathcal{P}_H^\kappa - q.s.$
- $\forall \mathbb{P} \in \mathcal{P}_H^\kappa$*, the process* $V^{\mathbb{P}}$ *defined below has paths of bounded variation* $\mathbb{P} - a.s.$

$$V_t^{\mathbb{P}} := Y_0 - Y_t - \int_0^t \widehat{F}_s(Y_s, Z_s)ds + \int_0^t Z_s dB_s, \ 0 \leq t \leq T, \ \mathbb{P} - a.s. \tag{1.2.30}$$

- *We have the following maximum condition for* $0 \leq t \leq T$

$$V_t^{\mathbb{P}} + k_t^{\mathbb{P},+} - k_t^{\mathbb{P},-} = \underset{\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'} \left[V_T^{\mathbb{P}'} + k_T^{\mathbb{P}',+} - k_T^{\mathbb{P}',-} \right], \ \mathbb{P}-a.s., \ \forall \mathbb{P} \in \mathcal{P}_H^{\kappa}. \quad (1.2.31)$$

- $L_t \leq Y_t \leq S_t$, $\mathcal{P}_H^{\kappa} - q.s.$

Theorem 11. *The superhedging and subhedging prices Y and $\widetilde{Y}$ are respectively the unique solution of the 2DRBSDE with terminal condition ξ, generator b, lower obstacle L, upper obstacle S in the sense of Definitions 6 and 7 respectively. The corresponding hedging strategies are then given by Z and $\widetilde{Z}$.*

Moreover, for any $\varepsilon > 0$ and for any $\mathbb{P}$, the following stopping times are ε-optimal after t for the seller and the buyer respectively

$$D_t^{1,\varepsilon,\mathbb{P}} := \inf \left\{ s \geq t, \ y_s^{\mathbb{P}} \geq S_s - \varepsilon, \ \mathbb{P}-a.s. \right\}, \ D_t^{2,\varepsilon} := \inf \left\{ s \geq t, \ Y_s \leq L_s + \varepsilon, \ \mathcal{P}_H^{\kappa} - q.s. \right\}.$$

Assumption (H)

We suppose that the following "min-max" property are satisfied. For any $\mathbb{P} \in \mathcal{P}_H^{\kappa}$

$$\underset{\tau \in \mathcal{T}_{t,T}}{\operatorname{ess\,inf}} \underset{\sigma \in \mathcal{T}_{t,T}}{\operatorname{ess\,sup}} \underset{\mathbb{P}' \in \mathcal{P}_H^{\kappa}(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'}[R_t(\tau,\sigma)] = \underset{\mathbb{P}' \in \mathcal{P}_H^{\kappa}(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \underset{\tau \in \mathcal{T}_{t,T}}{\operatorname{ess\,inf}} \underset{\sigma \in \mathcal{T}_{t,T}}{\operatorname{ess\,sup}} \mathbb{E}_t^{\mathbb{P}'}[R_t(\tau,\sigma)], \ \mathbb{P}-a.s. \quad (1.2.32)$$

$$\underset{\sigma \in \mathcal{T}_{t,T}}{\operatorname{ess\,sup}} \underset{\tau \in \mathcal{T}_{t,T}}{\operatorname{ess\,inf}} \underset{\mathbb{P}' \in \mathcal{P}_H^{\kappa}(t^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'}[R_t(\tau,\sigma)] = \underset{\mathbb{P}' \in \mathcal{P}_H^{\kappa}(t^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \underset{\sigma \in \mathcal{T}_{t,T}}{\operatorname{ess\,sup}} \underset{\tau \in \mathcal{T}_{t,T}}{\operatorname{ess\,inf}} \mathbb{E}_t^{\mathbb{P}'}[R_t(\tau,\sigma)], \ \mathbb{P}-a.s. \quad (1.2.33)$$

Theorem 12. *Let Assumption **(H)** hold. Let (Y,Z) (resp. $(\widetilde{Y},\widetilde{Z})$) be a solution to the 2DRBSDE in the sense of Definition 6 (resp. in the sense of Definition 7) with terminal condition ξ, generator g, lower obstacle L and upper obstacle S. Then we have for any $t \in [0,T]$*

$$\overline{V}_t = Y_t, \ \mathcal{P}_H^{\kappa} - q.s.$$
$$\underline{V}_t = \widetilde{Y}_t, \ \mathcal{P}_H^{\kappa} - q.s.$$

Moreover, unless $\mathcal{P}_H^{\kappa}$ is reduced to a singleton, we have $\overline{V} > \underline{V}$, $\mathcal{P}_H^{\kappa} - q.s.$

CHAPTER 2

SPDE with singular terminal condition

Contents

Introduction

We recall that a BDSDE is written as:

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r)\, dr + \int_t^T g(r, Y_r, Z_r)\, \overleftarrow{dB_r} - \int_t^T Z_r dW_r, \ 0 \leq t \leq T, \tag{2.0.1}$$

where B is a Brownian motion, independent of W, and $\overleftarrow{dB_r}$ is the backward Itô integral. These BSDE are connected with the following type of stochastic PDE: for $(t,x) \in [0,T] \times \mathbb{R}^d$

$$\begin{aligned} u(t,x) &= h(x) + \int_t^T [\mathcal{L}u(s,x) + f(s,x,u(s,x),(\nabla u\sigma)(s,x))]\, ds \\ &+ \int_t^T g(s,x,u(s,x),(\nabla u\sigma)(s,x)) \overleftarrow{dB_r}. \end{aligned} \tag{2.0.2}$$

The main goal of this chapter is to extend the results of [83] to BDSDE and to obtain a solution for a SPDE with singular terminal condition h. But before we have to prove existence and uniqueness of the solution of a BDSDE with monotone generator f. To our best knowlegde the closest result on this topic is in Aman [2]. Nevertheless we think that there is a lack in this paper (proof 4.2). Indeed for monotone BSDE ($g = 0$) the existence of a solution relies on the solvability of the BSDE:

$$Y_t = \xi + \int_t^T f(r, Y_r)\, dr - \int_t^T Z_r dW_r, \ 0 \leq t \leq T.$$

See among other the proof of Theorem 2.2 and Proposition 2.4 in [72]. To obtain a solution for this BSDE, the main trick is to truncate the coefficients with suitable truncation functions in order to have a bounded solution Y (see Proposition 2.2 in [18]). This can not be done for a general BDSDE. Indeed take for example ($\xi = f = 0$ and $g = 1$):

$$Y_t = \int_t^T \overleftarrow{dB_r} - \int_t^T Z_r dW_r = B_T - B_t, \ 0 \leq t \leq T,$$

with $Z = 0$. Thus in order to prove existence of a solution for (2.0.1), one can not directly follow the scheme of [72]. The first part of this paper is devoted to the existence of a solution for a monotone BDSDE (see Section 2.2) in the space $\mathcal{E}^2$. To realize this project we will restrict the class of functions f: they should satisfy a polynomial growth condition (as in [18]). Until now we do not know how to extend this to general growth condition as in [72] or [19].
The second goal of this work is to extend the results of [83] to the doubly stochastic framework. We will consider the generator $f(y) = -y|y|^q$ with $q \in \mathbb{R}_+^*$ and a real $\mathcal{F}_T^W$-measurable and non negative random variable ξ such that:

$$\mathbb{P}(\xi = +\infty) > 0. \tag{2.0.3}$$

And we want to find a solution to the following BDSDE:

$$Y_t = \xi - \int_t^T Y_s |Y_s|^q ds + \int_t^T g(s, Y_s, Z_s) \overleftarrow{dB_s} - \int_t^T Z_s dW_s. \tag{2.0.4}$$

The scheme to construct a solution is almost the same as in [83]. Let us emphasize one of main technical difficulties. If $g = 0$, we can use the conditional expectation w.r.t. $\mathcal{F}_t$ to withdraw the martingale part. If $g \neq 0$, this trick is useless and we have to be very careful when we want almost sure property of the solution. This BDSDE is connected with the stochastic PDE with terminal condition h: for any $0 \leq t \leq T$

$$\begin{aligned} u(t,x) &= h(x) + \int_t^T \left(\mathcal{L}u(s,x) - u(s,x)|u(s,x)|^q \right) ds \\ &\quad + \int_t^T g(s, u(s,x), \sigma(s,x)\nabla u(s,x)) \overleftarrow{dB_s}. \end{aligned} \tag{2.0.5}$$

If h is a smooth function we could use the result of Pardoux and Peng [74]. But here we will assume that $\mathcal{S} = \{h = +\infty\}$ is a closed non empty set and thus we will precise the notion of solution for (2.0.5) in this case. Roughly speaking we will show that there is a minimal solution u in the sense that u belongs to a Sobolev space and is a weak solution of the SPDE on any interval $[0, T - \delta]$, $\delta > 0$ and satisfies the terminal condition: $u(t, x)$ goes to $h(x)$ also in a weak sense as t goes to T.
The chapter is decomposed as follows. In the first section, we give the mathematical setting and our main contributions. In the next section, we study the existence and uniqueness of a class of monotone BDSDE. In Section 2.3 we construct a (super)solution for the BDSDE with singular terminal condition. In Section 2.4 we prove continuity at time T for this solution under sufficient conditions. Finally in the last part, we connect BDSDE and SPDE with a singularity at time T.

2.1 Setting and main results

Let us now precise our notations. W and B are independent Brownian motions defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}^k$ and $\mathbb{R}^m$. Let $\mathcal{N}$ denote the class of $\mathbb{P}$-null sets of $\mathcal{F}$. For each $t \in [0,T]$, we define

$$\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B$$

where for any process η, $\mathcal{F}_{s,t}^\eta = \sigma\{\eta_r - \eta_s;\ s \le r \le t\} \vee \mathcal{N}$, $\mathcal{F}_t^\eta = \mathcal{F}_{0,t}^\eta$. As in [74] we define the following filtration $(\mathcal{G}_t,\ t \in [0,T])$ by:

$$\mathcal{G}_t = \mathcal{F}_t^W \vee \mathcal{F}_{0,T}^B.$$

ξ is a $\mathcal{F}_T^W$-measurable and $\mathbb{R}^d$-valued random variable.
We define by $\mathbb{H}^p(0,T;\mathbb{R}^n)$ the set of (classes of $d\mathbb{P} \times dt$ a.e. equal) n dimensional jointly measurable random processes $(X_t,\ t \ge 0)$ which satisty:

1. $\mathbb{E}\left(\int_0^T |X_t|^2 dt\right)^{p/2} < +\infty$
2. X_t is $\mathcal{G}_t$-measurable for a.e. $t \in [0,T]$.

We denote similarly by $\mathbb{S}^p(0,T;\mathbb{R}^n)$ the set of continuous n dimensional random processes which satisfy:

1. $\mathbb{E}\left(\sup_{t\in[0,T]} |X_t|^p\right) < +\infty$
2. X_t is $\mathcal{G}_t$-measurable for any $t \in [0,T]$.

$\mathbb{B}^p(0,T)$ is the product $\mathbb{S}^p(0,T;\mathbb{R}^d) \times \mathbb{H}^p(0,T;\mathbb{R}^{d\times k})$. $(Y,Z) \in \mathcal{E}^p(0,T)$ if $(Y,Z) \in \mathbb{B}^p(0,T)$ and Y_t and Z_t are $\mathcal{F}_t$-measurable. Finally $C^{p,q}([0,T] \times \mathbb{R}^d; \mathbb{R})$ denotes the space of $\mathbb{R}$-valued functions defined on $[0,T] \times \mathbb{R}^d$ which are p-times continuously differentiable in $t \in [0,T]$ and q-times continuously differentiable in $x \in \mathbb{R}^d$. $C_b^{p,q}([0,T] \times \mathbb{R}^d; \mathbb{R})$ is the subspace of $C^{p,q}([0,T] \times \mathbb{R}^d; \mathbb{R})$ in which all functions have uniformly bounded partial derivatives; and $C_c^{p,q}([0,T] \times \mathbb{R}^d; \mathbb{R})$ the subspace of $C^{p,q}([0,T] \times \mathbb{R}^d; \mathbb{R})$ in which the functions have a compact support w.r.t. $x \in \mathbb{R}^d$.
Now we precise our assumptions on f and g. The functions f and g are defined on $[0,T] \times \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times k}$ with values respectively in $\mathbb{R}^d$ and $\mathbb{R}^{d\times m}$. Moreover we consider the following

Assumptions (A):

- The function $y \mapsto f(t,y,z)$ is continuous and there exists a constant μ such that for any (t,y,y',z) a.s.
$$\langle y - y', f(t,y,z) - f(t,y',z)\rangle \le \mu|y-y'|^2. \tag{A1}$$
- There exists K_f such that for any (t,y,z,z') a.s.
$$|f(t,y,z) - f(t,y,z')|^2 \le K_f|z-z'|^2. \tag{A2}$$
- There exists $C_f \ge 0$ and $p > 1$ such that
$$|f(t,y,z) - f(t,0,z)| \le C_f(1+|y|^p). \tag{A3}$$

- There exists a constant $K_g \geq 0$ and $0 < \varepsilon < 1$ such that for any (t, y, y', z, z') a.s.

$$|g(t,y,z) - g(t,y',z')|^2 \leq K_g|y-y'|^2 + \varepsilon|z-z'|^2. \tag{A4}$$

- Finally for any (t, y, z), $f(t, y, z)$ and $g(t, y, z)$ are $\mathcal{F}_t$ measurable with

$$\mathbb{E}\int_0^T \left(|f(t,0,0)|^2 + |g(t,0,0)|^2\right) dt < +\infty. \tag{A5}$$

Remember that from [74] if f also satisfies: there exists $\tilde{K}_f$ such that for any (t, y, y', z) a.s.

$$|f(t,y,z) - f(t,y',z)| \leq \tilde{K}_f|y-y'| \tag{2.1.6}$$

then there exists a unique solution $(Y, Z) \in \mathcal{E}^2(0, T)$ to the BDSDE (2.0.1). Note that (2.1.6) implies that

$$|f(t,y,z) - f(t,0,z)| \leq \tilde{K}_f|y|,$$

thus the growth assumption (A3) on f is satisfied with $p = 1$.
In Section 2.2 we will prove the following result.

Theorem 13. *Under Assumptions (A) and if the terminal condition ξ satisfies*

$$\mathbb{E}(|\xi|^2) < +\infty, \tag{2.1.7}$$

the BDSDE (2.0.1) has a unique solution $(Y, Z) \in \mathcal{E}^2(0, T)$.

Using the paper of Aman [2], this result can be extended to the L^p case: for $p \in (1, 2)$

$$\mathbb{E}\int_0^T (|f(t,0,0)|^p + |g(t,0,0)|^p)\, dt + \mathbb{E}(|\xi|^p) < +\infty.$$

There exists a unique solution in $\mathcal{E}^p(0, T)$.
The next sections are devoted to the singular case. The generator f will be supposed to be deterministic and given by: $f(y) = -y|y|^q$ for some $q > 0$. The aim is to prove existence of a solution for BDSDE (2.0.4) when the non negative random variable ξ satisfies (2.0.3). A possible extension of the notion of solution for a BDSDE with singular terminal condition could be the following (see Definition 1 in [83]).

Definition 8 (Solution of the BDSDE (2.0.4))**.** *Let $q > 0$ and ξ a $\mathcal{F}_T^W$-measurable non negative random variable satisfying condition (2.0.3). We say that the process (Y, Z) is a solution of the BDSDE (2.0.4) if (Y, Z) is such that (Y_t, Z_t) is $\mathcal{F}_t$-measurable and:*

(D1) for all $0 \leq s \leq t < T$: $Y_s = Y_t - \int_s^t Y_r|Y_r|^q dr + \int_s^t g(r, Y_r, Z_r)\overleftarrow{dB_r} - \int_s^t Z_r dW_r$;

(D2) for all $t \in [0, T[$, $\mathbb{E}\left(\sup_{0\leq s\leq t}|Y_s|^2 + \int_0^t \|Z_r\|^2 dr\right) < +\infty$;

(D3) $\mathbb{P}$-a.s. $\lim_{t\to T} Y_t = \xi$.

A solution is said non negative if a.s. for any $t \in [0, T]$, $Y_t \geq 0$.

To obtain an a priori estimate of the solution we will assume that $g(t, y, 0) = 0$ for any (t, y) a.s. This condition will ensure that our solutions will be non negative and bounded on any time interval $[0, T-\delta]$ with $\delta > 0$. Without this hypothesis, integrability of the solution would be more challenging. In Section 2.3, we will prove the following result.

Theorem 14. *There exists a process (Y, Z) satisfying Conditions (D1) and (D2) of Definition 8 and such that Y has a limit at time T with*

$$\lim_{t \to T} Y_t \geq \xi.$$

Moreover this solution is minimal: if $(\widetilde{Y}, \widetilde{Z})$ is a non negative solution of (2.0.4), then a.s. for any t, $\widetilde{Y}_t \geq Y_t$.

It means in particular that Y_t has a left limit at time T.
In general we are not able to prove that (D3) holds. As in [83], we give sufficient conditions for continuity and we prove it in the Markovian framework. Hence the first hypothesis on ξ is the following:

$$\xi = h(X_T), \tag{H1}$$

where h is a function defined on $\mathbb{R}^d$ with values in $\overline{\mathbb{R}^+}$ such that the set of singularity $\mathcal{S} = \{h = +\infty\}$ is closed; and where X_T is the value at $t = T$ of a diffusion process or more precisely the solution of a stochastic differential equation (in short SDE):

$$X_t = x + \int_0^t b(r, X_r)dr + \int_0^t \sigma(r, X_r)dW_r, \text{ for } t \in [0, T]. \tag{2.1.8}$$

We will always assume that b and σ are defined on $[0, T] \times \mathbb{R}^d$, with values respectively in $\mathbb{R}^d$ and $\mathbb{R}^{d \times k}$, are measurable w.r.t. the Borelian σ-algebras, and that there exists a constant $K > 0$ s.t. for all $t \in [0, T]$ and for all $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$:

1. Lipschitz condition:

$$|b(t, x) - b(t, y)| + \|\sigma(t, x) - \sigma(t, y)\| \leq K|x - y|; \tag{L}$$

2. Growth condition:

$$|b(t, x)| \leq K(1 + |x|) \text{ and } \|\sigma(t, x)\| \leq K(1 + |x|). \tag{G}$$

It is well known that under the previous assumptions, Equation (2.1.8) has a unique strong solution X.
We denote $\mathcal{R} = \mathbb{R}^d \setminus \mathcal{S}$. The second hypothesis on ξ is: for all compact set $\mathcal{K} \subset \mathcal{R} = \mathbb{R}^d \setminus \{h = +\infty\}$

$$h(X_T)\mathbf{1}_{\mathcal{K}}(X_T) \in \; L^1(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}). \tag{H2}$$

Unfortunately the above assumptions are not sufficient to prove continuity if $q \leq 2$. Thus we add the following conditions in order to use Malliavin calculus and to prove Equality (2.4.41).

1. The functions σ and b are bounded: there exists a constant K s.t.

$$\forall (t, x) \in [0, T] \times \mathbb{R}^d, \quad |b(t, x)| + \|\sigma(t, x)\| \leq K. \tag{B}$$

2. The second derivatives of $\sigma\sigma^*$ belongs to L^∞:
$$\frac{\partial^2 \sigma\sigma^*}{\partial x_i \partial x_j} \in L^\infty([0,T]\times\mathbb{R}^d). \tag{D}$$

3. $\sigma\sigma^*$ is uniformly elliptic, i.e. there exists $\lambda > 0$ s.t. for all $(t,x) \in [0,T]\times\mathbb{R}^d$:
$$\forall y \in \mathbb{R}^d,\ \sigma\sigma^*(t,x)y.y \geq \lambda |y|^2. \tag{E}$$

4. h is continuous from $\mathbb{R}^d$ to $\overline{\mathbb{R}_+}$ and:
$$\forall M \geq 0,\ h \text{ is a Lipschitz function on the set } \mathcal{O}_M = \{|h| \leq M\}. \tag{H3}$$

Theorem 15. *Under assumptions (H1), (H2) and (L) and if*

- *either $q > 2$ and (G);*
- *or (B), (D), (E) and (H3);*

the minimal non negative solution (Y,Z) of (2.0.4) satisfies (D3): a.s.
$$\lim_{t\to T} Y_t = \xi.$$

Finally in section 2.5, we show that this minimal solution (Y,Z) of (2.0.4) is connected to the minimal weak solution u of the SPDE (2.0.5). More precisely $X^{t,x}$ is the solution of the SDE (2.1.8) with initial condition x at time t and $(Y^{t,x},Z^{t,x})$ is the minimal solution of the BDSDE (2.0.4) with singular terminal condition $\xi = h(X^{t,x}_T)$.
Let us define the space $\mathcal{H}(0,T)$ as in [6]. We take the following weight function $\rho : \mathbb{R}^d \to \mathbb{R}$: for $\kappa > d$
$$\rho(x) = \frac{1}{(1+|x|)^\kappa}. \tag{2.1.9}$$

The constant κ will be fixed later. $\mathcal{H}(0,T)$ is the set of the random fields $\{u(t,x);\ 0 \leq t \leq T,\ x \in \mathbb{R}^d\}$ such that $u(t,x)$ is $\mathcal{F}^B_{t,T}$-measurable for each (t,x), u and $\sigma^*\nabla u$ belong to $L^2((0,T)\times\Omega\times\mathbb{R}^d;\ ds\otimes d\mathbb{P}\otimes\rho(x)dx)$. On $\mathcal{H}(0,T)$ we consider the following norm
$$\|u\|_2^2 = \mathbb{E}\int_0^T \int_{\mathbb{R}^d} \left(|u(s,x)|^2 + |(\sigma^*\nabla u)(s,x)|^2\right)\rho(x)dx.$$

Theorem 16. *The random field u defined by $u(t,x) = Y^{t,x}_t$ belongs to $\mathcal{H}(0,T-\delta)$ for any $\delta > 0$ and is a weak solution of the SPDE (2.0.5) on $[0,T-\delta]\times\mathbb{R}^d$. At time T, u satisfies a.s. $\liminf_{t\to T} u(t,x) \geq h(x)$.*
Moreover under the same assumptions of Theorem 15, for any function $\phi \in C^\infty_c(\mathbb{R}^d)$ with support included in $\mathcal{R}$, then
$$\lim_{t\to T} \mathbb{E}\left(\int_{\mathbb{R}^d} u(t,x)\phi(x)dx\right) = \int_{\mathbb{R}^d} h(x)\phi(x)dx.$$
Finally u is the minimal non negative solution of (2.0.5).

The almost sure continuity of u at time T is still an open question. In [83], this property is proved using viscosity solution arguments (relaxation of the boundary condition). Here we cannot do the same trick. This point will be investigated in further publications.
In the continuation, unimportant constants will be denoted by C.

2.2 Monotone BDSDE

As mentioned in the introduction and in the previous section, our first contribution is the extension of the result of Pardoux and Peng [74] with monotone condition (A1). We begin with the particular case where f does not depend on z and g is a given random field.

2.2.1 Case with $f(t,y,z) = f(t,y)$ and $g(t,y,z) = g_t$

In this special case assume that there exists a solution to the BDSDE:

$$Y_t = \xi + \int_t^T f(r, Y_r)\, dr + \int_t^T g_r \overleftarrow{dB_r} - \int_t^T Z_r dW_r,\ 0 \le t \le T. \tag{2.2.10}$$

Then we have

$$Y_t + \int_0^t g_r \overleftarrow{dB_r} = \xi + \int_0^T g_r \overleftarrow{dB_r} + \int_t^T f(r, Y_r)\, dr - \int_t^T Z_r dW_r,\ 0 \le t \le T.$$

Let us define:

$$U_t = Y_t + \int_0^t g_r \overleftarrow{dB_r}, \qquad \zeta = \xi + \int_0^T g_r \overleftarrow{dB_r},$$

and

$$\phi(t,y) = f\left(t, y - \int_0^t g_r \overleftarrow{dB_r}\right).$$

Then (U,Z) satisfies:

$$U_t = \zeta + \int_t^T \phi(r, U_r)\, dr - \int_t^T Z_r dW_r,\ 0 \le t \le T. \tag{2.2.11}$$

The terminal condition ζ is $\mathcal{G}_T$-measurable and the generator ϕ satisfies the following assumptions.

1. ϕ is continuous w.r.t. y and (A1) is true with the same constant μ.
2. From (A3), there exists $p > 1$ such that
$$|\phi(t,y)| \le h(t) + C_\phi(1 + |y|^p). \tag{2.2.12}$$
where $C_\phi = C_f 2^{p-1}$ and
$$h(t) = |f(t,0)| + 2^{p-1}\left|\int_0^t g_r \overleftarrow{dB_r}\right|^p.$$

On the solution (U,Z) we impose the following measurability constraints:

(M1). The process (U,Z) is adapted to the filtration $(\mathcal{G}_t,\ t \ge 0)$.

(M2). The random variable $U_t - \int_0^t g_r \overleftarrow{dB_r}$ is $\mathcal{F}_t$-measurable for any $0 \le t \le T$.

Let us assume the boundedness hypothesis on ξ, g and $f(t,0)$: there exists a constant $\gamma > 0$ such that a.s. for any $t \geq 0$,

$$|\xi| + |f(t,0)| + |g_t| \leq \gamma. \tag{2.2.13}$$

Hence for any $q > 1$

$$\mathbb{E}\left[|\zeta|^q + \left(\int_0^T |h(t)|^q dt\right)\right] < +\infty.$$

From [18] or [72] there exists a unique solution $(U,Z) \in \mathbb{B}^2(0,T)$ to the BSDE (2.2.11) such that (M1) holds and

$$\mathbb{E}\left[\sup_{t\in[0,T]} |U_t|^2 + \left(\int_0^T |Z_r|^2 dr\right)\right] < +\infty.$$

Theorem 3.6 in [18] also gives that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |U_t|^{2p} + \left(\int_0^T |Z_r|^2 dr\right)^p\right] < +\infty.$$

But we cannot derive directly from this result that (M2) is satisfied, that is $U_t - \int_0^t g_r \overleftarrow{dB_r}$ is $\mathcal{F}_t$-measurable for any $0 \leq t \leq T$. Therefore we follow the proof of Proposition 3.5 in [18] to prove the existence and uniqueness of the solution (U,Z) with the desired measurability conditions.

Proposition 3. *Under Assumptions* (A1), (A2), (A3) *and* (2.2.13), *there exists a unique solution $(U,Z) \in \mathbb{B}^2(0,T)$ to the BSDE* (2.2.11), *such that (M1) and (M2) hold.*

Proof. As written before, we sketch the proof of Proposition 3.5 in [18]. The details can be found in [18] and we just emphasize the main differences. For any $n \geq 1$, we define the following function:

$$\widetilde{\phi}_n(t,y) = \begin{cases} \phi(t,y) & \text{if } h(t) \leq n, \\ \frac{n}{|h(t)|}\phi(t,y) & \text{if } h(t) > n. \end{cases}$$

This function is continuous w.r.t. y and (A1) still holds. Moreover

$$|\widetilde{\phi}_n(t,y)| \leq (|h(t)| \wedge n) + C_\phi(1 + |y|^p).$$

Then as in [18], we define

$$\phi_n(t,.) = \rho_n * (\Theta_{q(n)+1}(\widetilde{\phi}_n(t,.))),$$

where

- $q(n) = \lceil e^{1/2}(n + 2C_\phi)\sqrt{1+T^2}\rceil + 1$, where $\lceil r \rceil$ stands for the integer part of r;
- Θ_n is a smooth function with values in $[0,1]$ such that $\Theta_n(u) = 1$ if $|u| \leq n$, $\Theta_n(u) = 0$ if $|u| \geq n+1$;
- $\rho_n(u) = n^k \rho(nu)$ with ρ a C^∞ non negative function with support equal to the unit ball and such that $\int \rho(u)du = 1$.

Since ζ is in $L^q(\Omega)$ (for any $q > 2p$) there exists a unique solution $(U^n, V^n) \in \mathbb{B}^q(0,T)$ to the BSDE (see Theorem 4.2 in [19] or Theorem 5.1 in [36]):

$$U_t^n = \zeta + \int_t^T \phi_n\left(r, U_r^n\right) dr - \int_t^T V_r^n dW_r,\ 0 \leq t \leq T. \tag{2.2.14}$$

Moreover for some constant K_p independent of n

$$\mathbb{E}\left[\sup_{t\in[0,T]}|U_t^n|^{2p}+\left(\int_0^T|V_r^n|^2dr\right)^p\right]\leq K_p\mathbb{E}\left[|\zeta|^{2p}+\left(\int_0^T(|h(r)|+2C_\phi)dr\right)^{2p}\right].$$

We have a strong convergence of the sequence (U^n,V^n) to (U,Z):

$$\lim_{n\to+\infty}\mathbb{E}\left[\sup_{t\in[0,T]}|U_t^n-U_t|^2+\left(\int_0^T|V_r^n-Z_r|^2dr\right)\right]=0.$$

And (U,Z) is the solution of BSDE (2.2.11) satisfying condition (M1) and $(U,Z)\in\mathbb{B}^{2p}(0,T)$. Now let us come to the measurability condition (M2). Recall that

$$\begin{aligned}\phi(t,y) &= f\left(t,y-\int_0^t g_r\overleftarrow{dB_r}\right),\\ h(t) &= |f(t,0)|+2^{p-1}\left|\int_0^t g_r\overleftarrow{dB_r}\right|^p,\end{aligned}$$

and the process $f(t,.)$ is $\mathcal{F}_t$-measurable. Hence for any y and t

$$\begin{aligned}\phi_n(t,y+\int_0^t g_r\overleftarrow{dB_r}) &= \rho_n*(\Theta_{q(n)+1}(\widetilde{\phi}_n(t,.)))(y+\int_0^t g_r\overleftarrow{dB_r})\\ &= \int\rho_n(z)\Theta_{q(n)+1}\left(\widetilde{\phi}_n\left(t,y-z+\int_0^t g_r\overleftarrow{dB_r}\right)\right)dz.\end{aligned}$$

Now

$$\widetilde{\phi}_n\left(t,x+\int_0^t g_r\overleftarrow{dB_r}\right)=\frac{n}{h(t)\vee n}f(t,x)=\frac{n}{h(t)\mathbf{1}_{h(t)\geq n}\vee n}f(t,x),$$

Thus $\phi_n(t,y+\int_0^t g_r\overleftarrow{dB_r})$ is measurable w.r.t. the σ-algebra $\mathcal{F}_t\vee\sigma(h(t)\mathbf{1}_{h(t)\geq n})$. Let us denote

$$\mathcal{H}^n=\sigma(h(t)\mathbf{1}_{h(t)\geq n},\ 0\leq t\leq T).$$

$\phi_n(t,y+\int_0^t g_r\overleftarrow{dB_r})$ is measurable w.r.t. the σ-algebra $\mathcal{F}_t^n=\mathcal{F}_t\vee\mathcal{H}^n$.
If we define

$$Y_t^n=U_t^n-\int_0^t g_r\overleftarrow{dB_r},$$

then

$$Y_t^n=\xi+\int_t^T g_r\overleftarrow{B_r}+\int_t^T\phi_n\left(r,Y_r^n+\int_0^r g_s\overleftarrow{B_s}\right)dr-\int_t^T V_r^ndW_r,\ 0\leq t\leq T.$$

We claim that Y_t^n is measurable w.r.t. $\mathcal{F}_t\vee\mathcal{H}^n$. Indeed let us recall that (U^n,V^n), solution of (2.2.14), is obtained via a fixed-point theorem. We define the map $\Psi:\mathbb{B}^2(0,T)\to\mathbb{B}^2(0,T)$ by: $(U,V)=\Psi(u,v)$ with

$$U_t=\zeta+\int_t^T\phi_n(r,u_r)\,dr-\int_t^T V_rdW_r,\ 0\leq t\leq T.$$

By classical arguments (see the details in [36], Theorem 2.1), Ψ is a contraction on $\mathbb{B}^2(0,T)$ (under suitable norms) and (U^n,V^n) is the fixed point of Ψ. Set $(U^{n,m},V^{n,m})$ for any $m\in\mathbb{N}$ as follows:

for any t, $(U^{n,0}_t, V^{n,0}_t) = (\int_0^t g_s \overleftarrow{B_s}, 0)$ and for any $m \geq 1$, $(U^{n,m}, V^{n,m}) = \Psi(U^{n,m-1}, V^{n,m-1})$. This sequence converges in $\mathbb{B}^2(0,T)$ to (U^n, V^n). Therefore Y^n is the limit in $\mathbb{S}^2(0,T)$ of $Y^{n,m}$ defined by: $Y^{n,0}_t = 0$ and

$$Y^{n,m}_t = \xi + \int_t^T g_r \overleftarrow{B_r} + \int_t^T \phi_n \left(r, Y^{n,m-1}_r + \int_0^r g_s \overleftarrow{B_s} \right) dr - \int_t^T V^{n,m}_r dW_r, \ 0 \leq t \leq T.$$

Now $Y^{n,0}_t$ is trivially $\mathcal{F}_t$ measurable and

$$\begin{aligned} Y^{n,m}_t &= \mathbb{E}\left[\xi + \int_t^T g_r \overleftarrow{B_r} \middle| \mathcal{G}_t \right] + \mathbb{E}\left[\int_t^T \phi_n \left(r, Y^{n,m-1}_r + \int_0^r g_s \overleftarrow{B_s} \right) dr \middle| \mathcal{G}_t \right] \\ &= \mathbb{E}\left[\xi + \int_t^T g_r \overleftarrow{B_r} \middle| \mathcal{G}_t \right] + \mathbb{E}\left[\Theta \middle| \mathcal{G}_t \right]. \end{aligned}$$

From [74] we know that the first term on the right hand side is $\mathcal{F}_t$ measurable. Assume that $Y^{n,m-1}_t$ is $\mathcal{F}_t \vee \mathcal{H}^n$ measurable. Since the same holds for $\phi_n \left(t, y + \int_0^t g_s \overleftarrow{B_s} \right)$, Θ depends only on $\mathcal{F}^W_T \vee \mathcal{F}^B_{t,T} \vee \mathcal{H}^n$. Thus there is no independence between $\mathcal{F}^B_t$ and $\mathcal{F}_t \vee \sigma(\Theta)$, but $Y^{n,m}_t$ depends on $\mathcal{F}^B_t$ only through $\mathcal{H}^n$. Hence $Y^{n,m}_t$ is $\mathcal{F}_t \vee \mathcal{H}^n$ measurable. Passing through the limit, we obtain the desired measurability condition on Y^n.
Now for any $m \in \mathbb{N}$, the sequence $(Y^n_t, \ n \geq m)$ depends on the σ-algebra

$$\mathcal{F}_t \vee \overline{\mathcal{H}_m} = \mathcal{F}_t \vee \left(\bigvee_{n \geq m} \mathcal{H}^n \right).$$

Passing through the limit, we obtain that the limit Y_t depends only on $\mathcal{F}_t \vee \mathcal{H}^\infty = \mathcal{F}_t \vee \bigcap_{m \in \mathbb{N}} \overline{\mathcal{H}_m}$. The next lemma shows that $\mathcal{H}^\infty \subset \mathcal{F}_0$. We deduce that Y_t is $\mathcal{F}_t$-measurable, which achieves the proof. □

Lemma 1. *The σ-algebra $\mathcal{H}^\infty$ is trivial: for every $A \in \mathcal{H}^\infty$, A or $A^c = \Omega \setminus A$ is negligible.*

Proof. Recall that f and g are supposed to be bounded by a constant γ and

$$h(t) = |f(t,0)| + 2^{p-1} \left| \int_0^t g_r \overleftarrow{dB_r} \right|^p.$$

Thus for any n

$$\mathbb{P}\left(\sup_{t \in [0,T]} h(t) \geq n \right) \leq \mathbb{P}\left(\sup_{t \in [0,T]} \left| \int_0^t g_r \overleftarrow{dB_r} \right|^p \geq 2^{1-p}(n-\gamma) \right).$$

The Burkholder-Davis-Gundy inequality shows that

$$\mathbb{E}\left(\sup_{t \in [0,T]} \left| \int_0^t g_r \overleftarrow{dB_r} \right|^p \right) \leq C_p \gamma^p T^{p/2}.$$

And by Markov inequality for $n > \gamma$

$$\mathbb{P}\left(\sup_{t \in [0,T]} \left| \int_0^t g_r \overleftarrow{dB_r} \right|^p \geq 2^{1-p}(n-\gamma) \right) \leq \frac{C_p \gamma^p T^{p/2}}{2^{1-p}(n-\gamma)}.$$

Note $\zeta = \sup_{t\in[0,T]} h(t)$. Now if $A \in \mathcal{H}^n$, then we have two cases: either the set $\{\zeta < n\}$ is included either in A or in A^c. And if $n \geq m$, then $\{\zeta < m\} \subset \{\zeta < n\}$. Hence if A is in $\bigvee_{n\geq m} \mathcal{H}^n$, then $\{\zeta < m\} \subset A$ or $A \cap \{\zeta < m\} = \emptyset$. And thus $\mathbb{P}(A) \wedge \mathbb{P}(A^c) \leq C/(m-\gamma)$ for any $m > \gamma$. Finally if $A \in \mathcal{H}^\infty = \bigcap_{m\in\mathbb{N}} \bigvee_{n\geq m} \mathcal{H}^n$, then $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$. $\square$

From the previous lemma, if we define

$$Y_t = U_t - \int_0^t g_r \overleftarrow{dB_r}$$

we obtain a solution (Y, Z) to the BDSDE:

$$Y_t = \xi + \int_t^T f(r, Y_r)\, dr + \int_t^T g_r \overleftarrow{dB_r} - \int_t^T Z_r dW_r,\ 0 \leq t \leq T.$$

From the boundedness assumption on g, we have:

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t|^2 + \left(\int_0^T |Z_r|^2 dr\right)\right] < +\infty.$$

From the previous proof Y_t is $\mathcal{F}_t$-measurable. Then using the same argument as in [74], the process Z_t is also $\mathcal{F}_t$-measurable. In other words $(Y, Z) \in \mathcal{E}^2(0, T)$.
Now we only assume that

$$\mathbb{E}\left[|\xi|^2 + \int_0^T (|f(t,0)|^2 + |g_t|^2) dt\right] < +\infty.$$

For any $n \in \mathbb{N}^*$ define Θ_n by

$$\Theta_n(y) = \begin{cases} y & \text{if } |y| \leq n, \\ n\dfrac{y}{|y|} & \text{if } |y| > n, \end{cases}$$

and $\xi^n = \Theta_n(\xi)$, $g_t^n = \Theta_n(g_t)$ and $f^n(t, y) = f(t, y) - f(t, 0) + \Theta_n(f(t, 0))$. Thus for a fixed n, there exists a solution (Y^n, Z^n) to the BDSDE (2.2.10) with ξ^n, f^n and g^n instead of ξ, f and g:

$$Y_t^n = \xi^n + \int_t^T f^n(r, Y_r^n)\, dr + \int_t^T g_r^n \overleftarrow{dB_r} - \int_t^T Z_r^n dW_r,\ 0 \leq t \leq T.$$

Define for any n and m

$$\Delta\xi = \xi^m - \xi^n, \quad \Delta f(t, y) = f^m(t, y) - f^n(t, y), \quad \Delta g_t = g_t^m - g_t^n,$$

and

$$\Delta Y_t = Y_t^m - Y_t^n, \quad \Delta Z_t = Z_t^m - Z_t^n.$$

From the Itô formula with $\alpha = 2\mu + 1$, we have:

$$\begin{aligned} e^{\alpha t}|\Delta Y_t|^2 + \int_t^T e^{\alpha r}|\Delta Z_r|^2 dr &= e^{\alpha T}|\Delta\xi|^2 + 2\int_t^T e^{\alpha s}\Delta Y_s(f^m(s, Y_s^m) - f^n(s, Y_s^n))ds \\ &- \int_t^T \alpha e^{\alpha s}|\Delta Y_s|^2 ds - 2\int_t^T e^{\alpha s}\Delta Y_s \Delta Z_s dW_s - 2\int_t^T e^{\alpha s}\Delta Y_s \Delta g_s \overleftarrow{dB_s} \\ &+ \int_t^T e^{\alpha s}|\Delta g_s|^2 ds. \end{aligned}$$

From assumption (A1) on f and $2|ab| \leq a^2 + b^2$, we obtain:

$$e^{\alpha t}|\Delta Y_t|^2 + \int_t^T e^{\alpha r}|\Delta Z_r|^2 dr \leq e^{\alpha T}|\Delta\xi|^2 + \int_t^T e^{\alpha s}|\Delta f(s,0)|^2 ds$$
$$-2\int_t^T e^{\alpha s}\Delta Y_s \Delta Z_s dW_s - 2\int_t^T e^{\alpha s}\Delta Y_s \Delta g_s \overleftarrow{dB_s} + \int_t^T e^{\alpha s}|\Delta g_s|^2 ds.$$

Using BDG inequality we deduce that there exists a constant C depending on α and T such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\Delta Y_t|^2 + \int_0^T |\Delta Z_r|^2 dr\right] \leq C\mathbb{E}\left[|\Delta\xi|^2 + \int_0^T |\Delta f(s,0)|^2 ds + \int_0^T |\Delta g_s|^2 ds\right].$$

Therefore (Y^n, Z^n) is a Cauchy sequence which converges to (Y,Z) and the limit process $(Y,Z) \in \mathcal{E}^2(0,T)$ satisfies the BDSDE (2.2.10).

Remark 1. *Can we assume a weaker growth condition on f ? Suppose that there exists a non decreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$|f(t,y)| \leq |f(t,0)| + \psi(|y|).$$

Using the same transformation, we have to control:

$$|\phi(t,y)| = |f(t, y + \int_0^t g_r \overleftarrow{dB_r})| \leq |f(t,0)| + \psi(|y + \int_0^t g_r \overleftarrow{dB_r}|).$$

If it is possible to find two functions ψ_1 and ψ_2 such that $\psi(y+z) \leq \psi_1(y) + \psi_2(z)$ and if $\psi_2(|\int_0^t g_r \overleftarrow{dB_r}|)$ belongs to $L^2(\Omega)$ for any bounded process g_t, it may be possible to obtain a solution with the desired properties to the BDSDE (2.0.1).
Until now we did not succeed to obtain an example which is not controlled by (A3) for some p large enough.

2.2.2 General case

The general case can be deduced from the previous one by a fixed-point argument. Let us construct the following sequence: $(Y^0, Z^0) = (0,0)$ and for $n \in \mathbb{N}$ and any $0 \leq t \leq T$

$$Y_t^{n+1} = \xi + \int_t^T f\left(r, Y_r^{n+1}, Z_r^n\right) dr + \int_t^T g\left(r, Y_r^n, Z_r^n\right) \overleftarrow{dB_r} - \int_t^T Z_r^{n+1} dW_r. \tag{2.2.15}$$

Indeed if

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^n|^2 + \left(\int_0^T |Z_r^n|^2 dr\right)\right] < +\infty$$

then from (A4) and (A5), the process $g_r^n = g\left(r, Y_r^n, Z_r^n\right)$ satisfies

$$\mathbb{E}\int_0^T |g_r^n|^2 dr < +\infty.$$

Moreover the process $f^n(r,0) = f(r,0,Z_r^n)$ verifies

$$\mathbb{E}\int_0^T |f(r,0,Z_r^n)|^2 ds \leq K_f^2 \mathbb{E}\int_0^T |Z_r^n|^2 dr + \mathbb{E}\int_0^T |f(r,0,0)|^2 dr < +\infty.$$

The previous section shows that (Y^{n+1}, Z^{n+1}) exists and satisfies (2.2.15) with

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^{n+1}|^2 + \left(\int_0^T |Z_r^{n+1}|^2 dr\right)\right] < +\infty.$$

Hence the sequence of processes (Y^n, Z^n) is well defined.
Now as before define for any n and m

$$\Delta Y_t^n = Y_t^{n+1} - Y_t^n, \quad \Delta Z_t^n = Z_t^{n+1} - Z_t^n, \quad \Delta g_t^n = g(t, Y_t^{n+1}, Z_t^{n+1}) - g(t, Y_t^n, Z_t^n).$$

From the Itô formula with $\alpha > 0$, we have:

$$\begin{aligned} e^{\alpha t}|\Delta Y_t^n|^2 + \int_t^T e^{\alpha r}|\Delta Z_r^n|^2 dr &= 2\int_t^T e^{\alpha s}\Delta Y_s^n (f(s, Y_s^{n+1}, Z_s^n) - f(s, Y_s^n, Z_s^{n-1}))ds \\ &- \int_t^T \alpha e^{\alpha s}|\Delta Y_s^n|^2 ds - 2\int_t^T e^{\alpha s}\Delta Y_s^n \Delta Z_s^n dW_s - 2\int_t^T e^{\alpha s}\Delta Y_s^n \Delta g_s^{n-1}\overleftarrow{dB_s} \\ &+ \int_t^T e^{\alpha s}|\Delta g_s^{n-1}|^2 ds. \end{aligned}$$

Using the Lipschitz assumption on g, we have

$$|\Delta g_s^{n-1}|^2 \le K_g|\Delta Y_s^{n-1}|^2 + \varepsilon|\Delta Z_s^{n-1}|^2.$$

And

$$\Delta Y_s^n (f(s, Y_s^{n+1}, Z_s^n) - f(s, Y_s^n, Z_s^{n-1})) \le \mu|\Delta Y_s^n|^2 + \sqrt{K_f}|\Delta Y_s^n||\Delta Z_s^{n-1}|.$$

Thus

$$\begin{aligned} e^{\alpha t}|\Delta Y_t^n|^2 + \int_t^T e^{\alpha r}|\Delta Z_r^n|^2 dr &\le (2\mu - \alpha)\int_t^T e^{\alpha s}|\Delta Y_s^n|^2 ds \\ &+2\sqrt{K_f}\int_t^T e^{\alpha s}|\Delta Y_s^n||\Delta Z_s^{n-1}|ds - 2\int_t^T e^{\alpha s}\Delta Y_s^n \Delta Z_s^n dW_s \\ &-2\int_t^T e^{\alpha s}\Delta Y_s^n \Delta g_s^{n-1}\overleftarrow{dB_s} + K_g\int_t^T e^{\alpha s}|\Delta Y_s^{n-1}|^2 ds + \varepsilon\int_t^T e^{\alpha s}|\Delta Z_s^{n-1}|^2 ds. \end{aligned} \tag{2.2.16}$$

Using the inequality $ab \le \eta a^2 + \frac{1}{\eta}b^2$, we have

$$2\sqrt{K_f}\int_t^T e^{\alpha s}|\Delta Y_s^n||\Delta Z_s^{n-1}|ds \le \eta K_f\int_t^T e^{\alpha s}|\Delta Y_s^n|^2 ds + \frac{1}{\eta}\int_t^T |\Delta Z_s^{n-1}|ds.$$

Therefore taking the expectation in (2.2.16) we deduce that

$$\begin{aligned} \mathbb{E}e^{\alpha t}|\Delta Y_t^n|^2 + \mathbb{E}\int_t^T e^{\alpha r}|\Delta Z_r^n|^2 dr &\le (2\mu + \eta K_f - \alpha)\mathbb{E}\int_t^T e^{\alpha s}|\Delta Y_s^n|^2 ds \\ &+ \left(\frac{1}{\eta} + \varepsilon\right)\mathbb{E}\int_t^T e^{\alpha s}|\Delta Z_s^{n-1}|ds + K_g\mathbb{E}\int_t^T e^{\alpha s}|\Delta Y_s^{n-1}|^2 ds. \end{aligned}$$

Take $t = 0$, $\eta = \frac{2}{1-\varepsilon}$ and $\alpha = 2\mu + \frac{2K_f}{1-\varepsilon} + \frac{2K_g}{1+\varepsilon}$ such that

$$\begin{aligned} &\mathbb{E}\int_0^T e^{\alpha r}|\Delta Z_r^n|^2 dr + \frac{2K_g}{1+\varepsilon}\mathbb{E}\int_0^T e^{\alpha s}|\Delta Y_s^n|^2 ds \\ &\quad\le \left(\frac{1+\varepsilon}{2}\right)\mathbb{E}\int_0^T e^{\alpha s}|\Delta Z_s^{n-1}|ds + \left(\frac{1+\varepsilon}{2}\right)\mathbb{E}\int_0^T \frac{2K_g}{1+\varepsilon}e^{\alpha s}|\Delta Y_s^{n-1}|^2 ds. \end{aligned} \tag{2.2.17}$$

Since $(1+\varepsilon)/2<1$, the sequence (Y^n,Z^n) is a Cauchy sequence in $L^2((0,T)\times\Omega)$ and converges to some process (Y,Z). Moreover by the BDG inequality we also obtain:

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}\left|\int_t^T e^{\alpha s}\Delta Y_s^n\Delta Z_s^n dW_s\right| &\le 4\mathbb{E}\left(\int_0^T e^{2\alpha s}|\Delta Y_s^n|^2|\Delta Z_s^n|^2 ds\right)^{1/2}\\ &\le \frac{1}{4}\mathbb{E}\left(\sup_{t\in[0,T]} e^{\alpha t}|\Delta Y_t^n|^2\right)+64\mathbb{E}\int_0^T e^{\alpha s}|\Delta Z_s^n|^2 ds,\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,T]}\left|\int_t^T e^{\alpha s}\Delta Y_s^n\Delta g_s^{n-1}\overleftarrow{dB_s}\right| \le 4\mathbb{E}\left(\int_0^T e^{2\alpha s}|\Delta Y_s^n|^2|\Delta g_s^{n-1}|^2 ds\right)^{1/2}\\ &\le \frac{1}{4}\mathbb{E}\left(\sup_{t\in[0,T]} e^{\alpha t}|\Delta Y_t^n|^2\right)+32\mathbb{E}\int_0^T e^{\alpha s}|\Delta g_s^{n-1}|^2 ds\\ &\le \frac{1}{4}\mathbb{E}\left(\sup_{t\in[0,T]} e^{\alpha t}|\Delta Y_t^n|^2\right)+32K_g\mathbb{E}\int_0^T e^{\alpha s}|\Delta Y_s^{n-1}|^2 ds+32\varepsilon\mathbb{E}\int_0^T e^{\alpha s}|\Delta Z_s^{n-1}|^2 ds.\end{aligned}$$

Coming back to (2.2.16) and using (2.2.17) we have for some constant C

$$\mathbb{E}\sup_{t\in[0,T]} e^{\alpha t}|\Delta Y_t^n|^2\le C\mathbb{E}\int_0^T e^{\alpha r}|\Delta Z_r^{n-1}|^2 dr+C\mathbb{E}\int_0^T e^{\alpha s}|\Delta Y_s^{n-1}|^2 ds.$$

We deduce also the convergence of Y^n to Y under this strong topology. Therefore (Y,Z) satisfies the general BDSDE:

$$Y_t=\xi+\int_t^T f(r,Y_r,Z_r)\,dr+\int_t^T g(r,Y_r,Z_r)\overleftarrow{dB_r}-\int_t^T Z_r dW_r,\ 0\le t\le T.$$

Hence we have proved Theorem 13.

2.2.3 Extension, comparison result

The extension of L^p solutions, $p\in(1,2)$, is done in Aman [2] (see also Theorem 1.4 in [74] for $p>2$). We just want here to recall the comparison principle for BDSDE (see [87], [62] or [37] on this topic). We will widely use the result in the next sections.

Proposition 4. *Assume that BDSDE (2.0.1) with data (f^1,g,ξ^1) and (f^2,g,ξ^2) have solutions (Y^1,Z^1) and (Y^2,Z^2) in $\mathcal{E}^2(0,T)$, respectively. The coefficient g satisfies (A4). If $\xi^1\le\xi^2$, a.s., and f^1 satisfies Assumptions (A1) and (A2), for all $t\in[0,T]$, $f^1(t,Y_t^2,Z_t^2)\le f^2(t,Y_t^2,Z_t^2)$, a.s. (resp. f^2 satisfies (A1) and (A2), for all $t\in[0,T]$, $f^1(t,Y_t^1,Z_t^1)\le f^2(t,Y_t^1,Z_t^1)$, a.s.), then we have $Y_t^1\le Y_t^2$, a.s., for all $t\in[0,T]$.*

Proof. The proof is almost the same as Lemma 3.1 in [62]. We define

$$(\hat{Y}_t,\hat{Z}_t)=(Y_t^1-Y_t^2,Z_t^1-Z_t^2),\quad \xi=\xi^1-\xi^2,$$

then $(\hat{Y}_t,\hat{Z}_t)$ satisfies the following BDSDE: for all $t\in[0,T]$,

$$\begin{aligned}\hat{Y}_t &= \hat{\xi}+\int_t^T\left[f^1\left(r,Y_r^1,Z_r^1\right)-f^2\left(r,Y_r^2,Z_r^2\right)\right]dr\\ &\quad+\int_t^T\left[g\left(r,Y_r^1,Z_r^1\right)-g\left(r,Y_r^2,Z_r^2\right)\right]\overleftarrow{dB_r}-\int_t^T\hat{Z}_r dW_r.\end{aligned}$$

We apply Itô's formula to $(\hat{Y}_t^+)^2$:

$$(\hat{Y}_t^+)^2 + \int_t^T \mathbf{1}_{\hat{Y}_r>0}|\hat{Z}_r|^2 dr \leq (\hat{\xi}^+)^2 + 2\int_t^T \hat{Y}_r^+ \left[f^1\left(r, Y_r^1, Z_r^1\right) - f^2\left(r, Y_r^2, Z_r^2\right)\right] dr$$
$$+2\int_t^T \hat{Y}_r^+ \left[g\left(r, Y_r^1, Z_r^1\right) - g\left(r, Y_r^2, Z_r^2\right)\right] \overleftarrow{dB_r}$$
$$-2\int_t^T \hat{Y}_r^+ \hat{Z}_r dW_r + \int_t^T \mathbf{1}_{\hat{Y}_r>0}|g(r, Y_r^1, Z_r^1) - g(r, Y_r^2, Z_r^2)|^2 dr.$$

Now from (A1) and (A2)

$$\begin{aligned}
&\hat{Y}_r^+ \left[f^1\left(r, Y_r^1, Z_r^1\right) - f^2\left(r, Y_r^2, Z_r^2\right)\right] \\
&= \hat{Y}_r^+ \left[f^1\left(r, Y_r^1, Z_r^1\right) - f^1\left(r, Y_r^2, Z_r^2\right)\right] + \hat{Y}_r^+ \left[f^1\left(r, Y_r^2, Z_r^2\right) - f^2\left(r, Y_r^2, Z_r^2\right)\right] \\
&\leq \mu(\hat{Y}_r^+)^2 + K_f \hat{Y}_r^+ |\hat{Z}_r|.
\end{aligned}$$

The rest of the proof is exactly the same as in [62]. Using Gronwall's Lemma, we deduce that $\mathbb{E}(\hat{Y}_t^+)^2 = 0$ for any $t \in [0,T]$. □

2.3 Singular terminal condition, construction of a minimal solution

From now on we assume that the terminal condition ξ satisfies the property (2.0.3):

$$\mathbb{P}(\xi \geq 0) = 1 \quad \text{and} \quad \mathbb{P}(\xi = +\infty) > 0.$$

For $q > 0$, let us consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(y) = -y|y|^q$. f is continuous and monotone, i.e. satisfies Condition (A1) with $\mu = 0$: for all $(y, y') \in \mathbb{R}^2$:

$$(y - y')(f(y) - f(y')) \leq 0.$$

Condition (A3) is also satisfied with $p = q+1$. We also consider a function $g : [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ and we assume that Condition (A4) holds.

2.3.1 Approximation

For every $n \in \mathbb{N}^*$, we introduce $\xi_n = \xi \wedge n$. ξ_n belongs to $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. We apply Theorem 13 with ξ_n as the final data, and we build a sequence of random processes $(Y^n, Z^n) \in \mathcal{E}^2(0,T)$ which satisfy (2.0.4).
From Proposition 4, if $n \leq m$, $0 \leq \xi_n \leq \xi_m \leq m$, which implies for all t in $[0,T]$, a.s.,

$$\Xi_t^0 \leq Y_t^n \leq Y_t^m \leq \Xi_t^m. \tag{2.3.18}$$

Here Ξ^k is the first component of the unique solution (Ξ^k, Θ^k) in $\mathcal{E}^2(0,T)$ of (2.0.4) with a deterministic terminal condition k. In order to have explicit and useful bound on Y^m we will assume that $g(t,y,0) = 0$ for any (t,y) a.s. In this case for $m \geq 1$,

$$\Xi_t^0 = 0, \qquad \Xi_t^m = \left(\frac{1}{q(T-t) + \frac{1}{m^q}}\right)^{\frac{1}{q}}.$$

We define the progressively measurable $\mathbb{R}$-valued process Y, as the increasing limit of the sequence $(Y_t^n)_{n\geq 1}$:

$$\forall t \in [0,T],\ Y_t = \lim_{n\to+\infty} Y_t^n. \tag{2.3.19}$$

Then we obtain

$$\forall 0 \leq t \leq T, \quad 0 \leq Y_t \leq \left(\frac{1}{q(T-t)}\right)^{\frac{1}{q}}. \tag{2.3.20}$$

In particular Y is finite on the interval $[0,T[$ and bounded on $[0,T-\delta]$ for all $\delta > 0$.
Here we will prove the first part of Theorem 14, that is (Y,Z) satisfies properties (D1) and (D2) of the definition 8. Moreover we will obtain that there exists a constant κ, depending on g, s.t. for all $t \in [0,T[$,

$$\mathbb{E}\int_0^t \|Z_r\|^2 dr \leq \frac{\kappa}{(q(T-t))^{\frac{2}{q}}}, \tag{2.3.21}$$

Proof. Let $\delta > 0$ and $s \in [0,T-\delta]$. For all $0 \leq t \leq s$, Itô's formula leads to the equality:

$$\begin{aligned}
|Y_t^n - Y_t^m|^2 &+ \int_t^s \|Z_r^n - Z_r^m\|^2 dr = |Y_s^n - Y_s^m|^2 - 2\int_t^s (Y_r^n - Y_r^m)(Z_r^n - Z_r^m)dW_r \\
&+2\int_t^s (Y_r^n - Y_r^m)\left(f(Y_r^n) - f(Y_r^m)\right)dr \\
&+2\int_t^s (Y_r^n - Y_r^m)(g(r,Y_r^n,Z_r^n) - g(r,Y_r^m,Z_r^m))\overleftarrow{dB_r} \\
&+\int_t^s |g(r,Y_r^n,Z_r^n) - g(r,Y_r^m,Z_r^m)|^2 dr \\
&\leq |Y_s^n - Y_s^m|^2 + K\int_t^s |Y_r^n - Y_r^m|^2 dr + \varepsilon\int_t^s |Z_r^n - Z_r^m|^2 dr \\
&-2\int_t^s (Y_r^n - Y_r^m)(Z_r^n - Z_r^m)dW_r \\
&+2\int_t^s (Y_r^n - Y_r^m)(g(r,Y_r^n,Z_r^n) - g(r,Y_r^m,Z_r^m))\overleftarrow{dB_r}
\end{aligned}$$

from the monotonicity of f (Inequality (A1)) and the Lipschtiz property of g (Inequality (A4)). From the properties (A4) and since $(Y,Z) \in \mathcal{E}^2$, we have:

$$\mathbb{E}\left(\int_t^s (Y_r^n - Y_r^m)(Z_r^n - Z_r^m)dW_r\right) = 0,$$

$$\mathbb{E}\left(\int_t^s (Y_r^n - Y_r^m)(g(r,Y_r^n,Z_r^n) - g(r,Y_r^m,Z_r^m))\overleftarrow{dB_r}\right) = 0.$$

From the Burkholder-Davis-Gundy inequality, we deduce the existence of a universal constant C with:

$$\mathbb{E}\left(\sup_{0\leq t\leq s}|Y_t^n - Y_t^m|^2 + \int_0^s \|Z_r^n - Z_r^m\|^2 dr\right) \leq C\,\mathbb{E}\left(|Y_s^n - Y_s^m|^2\right). \tag{2.3.22}$$

From the estimate (2.3.20), for $s \leq T-\delta$, $Y_s^n \leq \frac{1}{(q\delta)^{1/q}}$ and $Y_s \leq \frac{1}{(q\delta)^{1/q}}$. Since Y_s^n converges to Y_s a.s., the dominated convergence theorem and the previous inequality (2.3.22) imply:

1. for all $\delta > 0$, $(Z^n)_{n\geq 1}$ is a Cauchy sequence in $L^2(\Omega \times [0, T-\delta]; \mathbb{R}^d)$, and converges to $Z \in L^2(\Omega \times [0, T-\delta]; \mathbb{R}^d)$,
2. $(Y^n)_{n\geq 1}$ converges to Y uniformly in mean-square on the interval $[0, T-\delta]$, in particular Y is continuous on $[0, T)$,
3. (Y, Z) satisfies Equation (2.0.4) on $[0, T)$.

Since Y_t is smaller than $1/(q(T-t))^{1/q}$ by (2.3.20), and since $Z \in L^2(\Omega \times [0, T-\delta]; \mathbb{R}^d)$, applying the Itô formula to $|Y|^2$, with $s < T$ and $0 \leq t \leq s$, we obtain:

$$\begin{aligned}
|Y_t|^2 + \int_t^s \|Z_r\|^2 dr &= |Y_s|^2 - 2\int_t^s Y_r Z_r dW_r + 2\int_t^s Y_r f(Y_r) dr \\
&\quad + 2\int_t^s Y_r g(r, Y_r, Z_r) \overleftarrow{dB_r} + \int_t^s |g(r, Y_r, Z_r)|^2 dr \\
&\leq \frac{1}{(q(T-s))^{\frac{2}{q}}} - 2\int_t^s Y_r Z_r dW_r + 2\int_t^s Y_r g(r, Y_r, Z_r) \overleftarrow{dB_r} \\
&\quad + K\int_t^s |Y_r|^2 dr + \varepsilon \int_t^s |Z_r|^2 dr,
\end{aligned}$$

again thanks to Inequalities (A1) and (A4). From (2.3.20), since $Z \in L^2([0, s] \times \Omega)$, we have:

$$\mathbb{E}\int_t^s Y_r Z_r dW_r = \mathbb{E}\int_t^s Y_r g(r, Y_r, Z_r) \overleftarrow{dB_r} = 0.$$

Therefore, we deduce that there exists a constant κ depending on T, K and ε such that :

$$\mathbb{E}\int_0^s \|Z_r\|^2 dr \leq \frac{\kappa}{(q(T-s))^{\frac{2}{q}}}.$$

□

Remark that if g is equal to zero, κ is equal to one. And in general

$$\kappa = \frac{1}{1-\varepsilon}(1 + KT).$$

We want to establish the following statement which completes Inequality (2.3.21).

Proposition 5. *The next inequality is a sharper estimation on Z:*

$$\mathbb{E}\int_0^T (T-s)^{2/q} \|Z_s\|^2 ds \leq \frac{8 + KT}{1-\varepsilon} \left(\frac{1}{q}\right)^{2/q}. \tag{2.3.23}$$

The constants K and ε are given by the assumption (A4).

Proof. First suppose there exists a constant $\alpha > 0$ such that $\mathbb{P}$-a.s. $\xi \geq \alpha$. In this case, by comparison, for all integer n and all $t \in [0, T]$:

$$Y_t^n \geq \left(\frac{1}{qT + 1/\alpha^q}\right)^{1/q} > 0.$$

Let $\delta > 0$ and $\theta : \mathbb{R} \to \mathbb{R}$, $\theta_q : \mathbb{R} \to \mathbb{R}$ defined by:

$$\left\{ \begin{array}{lcll} \theta(x) & = & \sqrt{x} & \text{on } [\delta, +\infty[, \\ \theta(x) & = & 0 & \text{on }]-\infty, 0], \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{lcll} \theta_q(x) & = & x^{\frac{1}{2q}} & \text{on } [\delta, +\infty[, \\ \theta_q(x) & = & 0 & \text{on }]-\infty, 0], \end{array} \right.$$

and such that θ and θ_q are non-negative, non-decreasing and in respectively $C^2(\mathbb{R})$ and $C^1(\mathbb{R})$. We apply the Itô formula on $[0, T-\delta]$ to the function $\theta_q(T-t)\theta(Y_t^n)$, with $\delta < (qT + 1/\alpha^q)^{-1/q}$:

$$\begin{aligned} \theta_q(\delta)\theta(Y^n_{T-\delta}) - \theta_q(T)\theta(Y^n_0) &= \frac{1}{2}\int_0^{T-\delta} (T-s)^{1/2q}(Y_s^n)^{1/2}\left((Y_s^n)^q - \frac{1}{q(T-s)}\right) ds \\ &+ \frac{1}{2}\int_0^{T-\delta} (T-s)^{1/2q}(Y_s^n)^{-1/2} Z_s^n dW_s \\ &- \frac{1}{2}\int_0^{T-\delta} (T-s)^{1/2q}(Y_s^n)^{-1/2} g(s, Y_s^n, Z_s^n)\overleftarrow{dB_s} \\ &- \frac{1}{8}\int_0^{T-\delta} (T-s)^{1/2q}\frac{\|Z_s^n\|^2 - g(s, Y_s^n, Z_s^n)^2}{(Y_s^n)^{3/2}} ds. \end{aligned}$$

If we define

$$\Psi_s^n = \frac{\|Z_s^n\|^2 - g(s, Y_s^n, Z_s^n)^2}{(Y_s^n)^{3/2}},$$

we have

$$\begin{aligned} \frac{1}{8}\int_0^{T-\delta} (T-s)^{1/2q}\Psi_s^n ds &\le T^{1/2q}\theta(Y_0^n) + \frac{1}{2}\int_0^{T-\delta} (T-s)^{1/2q}(Y_s^n)^{-1/2} Z_s^n dW_s \\ &+\frac{1}{2}\int_0^{T-\delta} (T-s)^{1/2q}(Y_s^n)^{1/2}\left((Y_s^n)^q - \frac{1}{q(T-s)}\right) ds \\ &-\frac{1}{2}\int_0^{T-\delta} (T-s)^{1/2q}(Y_s^n)^{-1/2} g(s, Y_s^n, Z_s^n)\overleftarrow{dB_s} \end{aligned}$$

and since $Y_s^n \le 1/(q(T-s))^{1/q}$ and $T^{1/q}Y_0^n \le q^{-1/q}$, taking the expectation we obtain:

$$\frac{1}{8}\mathbb{E}\int_0^{T-\delta} (T-s)^{1/2q}\Psi_s^n ds \le \theta_q(T)\theta(Y_0^n) \le (1/q)^{1/2q},$$

that is for all n and all $\delta > 0$:

$$\mathbb{E}\int_0^{T-\delta} (T-s)^{1/2q}\Psi_s^n ds \le 8(1/q)^{1/2q}.$$

Using the assumption (A4) on g, we have

$$(1-\varepsilon)\frac{\|Z_s^n\|^2}{(Y_s^n)^{3/2}} \le \Psi_s^n + K(Y_s^n)^{1/2},$$

which implies

$$\begin{aligned} (1-\varepsilon)\mathbb{E}\int_0^{T-\delta} (T-s)^{1/2q}\frac{\|Z_s^n\|^2}{(Y_s^n)^{3/2}} ds &\le 8(1/q)^{1/2q} + K\mathbb{E}\int_0^{T-\delta} (T-s)^{1/2q}(Y_s^n)^{1/2} ds \\ \le 8(1/q)^{1/2q} + K\mathbb{E}\int_0^{T-\delta} (T-s)^{1/2q}&\left(\frac{1}{q(T-s)}\right)^{1/(2q)} ds \le (1/q)^{1/2q}(8 + KT). \end{aligned}$$

Now, since $1/Y_s^n \geq (q(T-s))^{1/q}$, letting $\delta \to 0$ and with the Fatou lemma, we deduce that

$$\mathbb{E}\int_0^T (T-s)^{2/q}\|Z_s\|^2 ds \leq \frac{8+KT}{1-\varepsilon}(1/q)^{2/q}.$$

Now we come back to the case $\xi \geq 0$. We can not apply the Itô formula because we do not have any positive lower bound for Y^n. We will approach Y^n in the following way. We define for $n \geq 1$ and $m \geq 1$, $\xi^{n,m}$ by:

$$\xi^{n,m} = (\xi \wedge n) \vee \frac{1}{m}.$$

This random variable is in L^2 and is greater or equal to $1/m$ a.s. The BSDE (2.0.4), with $\xi^{n,m}$ as terminal condition, has a unique solution $(\widetilde{Y}^{n,m}, \widetilde{Z}^{n,m})$. It is immediate that if $m \leq m'$ and $n \leq n'$ then:

$$\widetilde{Y}^{n,m'} \leq \widetilde{Y}^{n',m}.$$

As for the sequence Y^n, we can define $\widetilde{Y}^m$ as the limit when n grows to $+\infty$ of $\widetilde{Y}^{n,m}$. This limit $\widetilde{Y}^m$ is greater than $Y = \lim_{n\to+\infty} Y^n$. But for any m and n, for $t \in [0,T]$:

$$\begin{aligned}
\left|\widetilde{Y}_t^{n,m} - Y_t^n\right|^2 &= |\xi^{n,m} - \xi^n|^2 - 2\int_t^T \left[\widetilde{Y}_r^{n,m} - Y_r^n\right]\left[\left(\widetilde{Y}_r^{n,m}\right)^{q+1} - (Y_r^n)^{q+1}\right] dr \\
&\quad -2\int_t^T \left[\widetilde{Y}_r^{n,m} - Y_r^n\right]\left[\widetilde{Z}_r^{n,m} - Z_r^n\right] dW_r - \int_t^T \left[\widetilde{Z}_r^{n,m} - Z_r^n\right]^2 dr \\
&\quad +2\int_t^T \left[\widetilde{Y}_r^{n,m} - Y_r^n\right]\left[g(r,\widetilde{Y}_r^{n,m},\widetilde{Z}_r^{n,m}) - g(r,Y_r^n,Z_r^n)\right] \overleftarrow{dB_r} \\
&\quad +\int_t^T \left[g(r,\widetilde{Y}_r^{n,m},\widetilde{Z}_r^{n,m}) - g(r,Y_r^n,Z_r^n)\right]^2 dr \\
&\leq |\xi^{n,m} - \xi^n|^2 - 2\int_t^T \left[\widetilde{Y}_r^{n,m} - Y_r^n\right]\left[\widetilde{Z}_r^{n,m} - Z_r^n\right] dW_r \\
&\quad +2\int_t^T \left[\widetilde{Y}_r^{n,m} - Y_r^n\right]\left[g(r,\widetilde{Y}_r^{n,m},\widetilde{Z}_r^{n,m}) - g(r,Y_r^n,Z_r^n)\right] \overleftarrow{dB_r} \\
&\quad -(1-\varepsilon)\int_t^T \left[\widetilde{Z}_r^{n,m} - Z_r^n\right]^2 dr + (1+K)\int_t^T \left[\widetilde{Y}_r^{n,m} - Y_r^n\right]^2 dr
\end{aligned} \tag{2.3.24}$$

and taking the expectation:

$$\mathbb{E}\left|\widetilde{Y}_t^{n,m} - Y_t^n\right|^2 \leq \mathbb{E}|\xi^{n,m} - \xi^n|^2 + (1+K)\int_t^T \mathbb{E}\left[\widetilde{Y}_r^{n,m} - Y_r^n\right]^2 dr.$$

Gronwall lemma shows that for any $t \in [0,T]$:

$$\mathbb{E}\left|\widetilde{Y}_t^{n,m} - Y_t^n\right|^2 \leq e^{(1+K)T}\mathbb{E}|\xi^{n,m} - \xi^n|^2 \leq e^{(1+K)T}\frac{1}{m^2}. \tag{2.3.25}$$

To conclude we fix $\delta > 0$ and we apply the Itô formula to the process $(T-.)^{2/q}\left|\widetilde{Y}^{n,m} - Y^n\right|^2$. This leads to the inequality:

$$\begin{aligned}
(1-\varepsilon)\mathbb{E}\int_0^{T-\delta} (T-r)^{2/q}\left\|\widetilde{Z}_r^{n,m} - Z_r^n\right\|^2 dr \quad &\leq \quad \frac{2}{q}\mathbb{E}\int_0^{T-\delta} (T-s)^{(2/q)-1}\left|\widetilde{Y}_s^{n,m} - Y_s^n\right|^2 ds \\
&\quad +(\delta)^{2/q}\mathbb{E}\left|\widetilde{Y}_{T-\delta}^{n,m} - Y_{T-\delta}^n\right|^2 \\
&\quad +K\mathbb{E}\int_0^{T-\delta} (T-r)^{2/q}\left(\widetilde{Y}_r^{n,m} - Y_r^n\right)^2 dr.
\end{aligned}$$

Let δ go to 0 in the previous inequality. We can do that because $(T-.)^{(2/q)-1}$ is integrable on the interval $[0,T]$ and because of (2.3.25). Finally we have

$$\begin{aligned}(1-\varepsilon)\mathbb{E}\int_0^T (T-r)^{2/q}\left\|\widetilde{Z}_r^{n,m}-Z_r^n\right\|^2 dr &\leq \frac{e^{(1+K)T}}{m^2}\left[\frac{2}{q}\int_0^T (T-s)^{(2/q)-1}ds + K\int_0^T (T-s)^{(2/q)}ds\right] \\ &= \frac{T^{2/q}e^{(1+K)T}}{m^2}\left(1+\frac{KT}{1+2/q}\right).\end{aligned}$$

Therefore, for all $\eta > 0$:

$$\begin{aligned}\mathbb{E}\int_0^T (T-r)^{2/q}\left\|Z_r^n\right\|^2 dr &\leq (1+\eta)\mathbb{E}\int_0^T (T-r)^{2/q}\left\|\widetilde{Z}_r^{n,m}\right\|^2 dr \\ &+ (1+\frac{1}{\eta})\mathbb{E}\int_0^T (T-r)^{2/q}\left\|\widetilde{Z}_r^{n,m}-Z_r^n\right\|^2 dr \\ &\leq (1+\eta)\frac{8+KT}{1-\varepsilon}(1/q)^{2/q} + (1+\frac{1}{\eta})\frac{T^{2/q}e^{(1+K)T}}{m^2(1-\varepsilon)}\left(1+\frac{KT}{1+2/q}\right).\end{aligned}$$

We have applied the previous result to $\widetilde{Z}^{n,m}$. Now we let first m go to $+\infty$ and then η go to 0, we have:

$$\mathbb{E}\int_0^T (T-r)^{2/q}\left\|Z_r^n\right\|^2 dr \leq \frac{8+KT}{1-\varepsilon}(1/q)^{2/q}.$$

The result follows by letting finally n go to ∞ and this achieves the proof of the proposition. □

2.3.2 Existence of a limit at time T

From now, the process Y is continuous on $[0,T[$ and we define $Y_T = \xi$. The main difficulty will be to prove the continuity at time T. It is easy to show that:

$$\xi \leq \liminf_{t\to T} Y_t. \tag{2.3.26}$$

Indeed, for all $n \geq 1$ and all $t \in [0,T]$, $Y_t^n \leq Y_t$, therefore:

$$\xi \wedge n = \liminf_{t\to T} Y_t^n \leq \liminf_{t\to T} Y_t.$$

Thus, Y is lower semi-continuous on $[0,T]$ (this is clear since Y is the supremum of continuous functions). But now we will show that Y has a limit on the left at time T. We will distinguich the case when ξ is greater than a positive constant from the case ξ non-negative. This will complete the proof of Theorem 14.

2.3.2.1 The case ξ bounded away from zero.

We can show that Y has a limit on the left at T by using Itô's formula applied to the process $1/(Y^n)^q$. Suppose there exists a real $\alpha > 0$ such that $\xi \geq \alpha > 0$, $\mathbb{P}$-a.s. Then from Proposition 4 (and since $g(t,y,0)=0$), for every $n \in \mathbb{N}^*$ and every $0 \leq t \leq T$:

$$n \geq Y_t^n \geq \left(\frac{1}{q(T-t)+1/\alpha^q}\right)^{1/q} \geq \left(\frac{1}{qT+1/\alpha^q}\right)^{1/q} > 0.$$

By the Itô formula

$$
\begin{aligned}
\frac{1}{(Y_t^n)^q} &= \frac{1}{(\xi \wedge n)^q} + q(T-t) - \frac{q(q+1)}{2} \int_t^T \frac{\|Z_s^n\|^2}{(Y_s^n)^{q+2}} ds + \int_t^T \frac{qZ_s^n}{(Y_s^n)^{q+1}} dW_s \\
&\quad -q \int_t^T \frac{g(s,Y_s^n,Z_s^n)}{(Y_s^n)^{1+q}} \overleftarrow{dB_s} + \frac{q(q+1)}{2} \int_t^T \frac{(g(s,Y_s^n,Z_s^n))^2}{(Y_s^n)^{2+q}} ds \\
&= \frac{1}{(\xi \wedge n)^q} + q(T-t) + \int_t^T \frac{qZ_s^n}{(Y_s^n)^{q+1}} dW_s \\
&\quad -q \int_t^T \frac{g(s,Y_s^n,Z_s^n)}{(Y_s^n)^{1+q}} \overleftarrow{dB_s} - \frac{q(q+1)}{2} \int_t^T \Psi_s^n ds,
\end{aligned} \tag{2.3.27}
$$

where

$$
\Psi_s^n = \frac{\|Z_s^n\|^2 - (g(s,Y_s^n,Z_s^n))^2}{(Y_s^n)^{q+2}}.
$$

Or for any $t \in [0,T]$:

$$
\mathbb{E}\frac{1}{(Y_t^n)^q} = \mathbb{E}\left(\frac{1}{(\xi \wedge n)^q}\right) + q(T-t) - \frac{q(q+1)}{2}\mathbb{E}\int_t^T \Psi_s^n ds.
$$

This shows that

$$
\sup_{n\in\mathbb{N}} \mathbb{E}\int_0^T \Psi_s^n ds < +\infty. \tag{2.3.28}
$$

From the assumption on g, we have

$$
(1-\varepsilon)\frac{\|Z_s^n\|^2}{(Y_s^n)^{q+2}} - K\frac{1}{(Y_s^n)^q} \le \Psi_s^n \Rightarrow (1-\varepsilon)\frac{\|Z_s^n\|^2}{(Y_s^n)^{q+2}} \le \Psi_s^n + K\frac{1}{(Y_s^n)^q}. \tag{2.3.29}
$$

Form this inequality and Inequality (2.3.28) we deduce that

$$
\sup_{n\in\mathbb{N}} \mathbb{E}\int_0^T \left(\frac{\|Z_s^n\|^2}{(Y_s^n)^{q+2}} + \frac{(g(s,Y_s^n,Z_s^n))^2}{(Y_s^n)^{2+q}}\right) ds < +\infty.
$$

Hence the two sequences

$$
\int_t^T \frac{qZ_s^n}{(Y_s^n)^{q+1}} dW_s \text{ and } \int_t^T \frac{g(s,Y_s^n,Z_s^n)}{(Y_s^n)^{1+q}} \overleftarrow{dB_s}
$$

converge weakly in L^2 to some stochastic integrals (the proof is classical and uses Mazur's lemma (see [95], chapter V.1, Theorem 2, for example)):

$$
\int_t^T U_s dW_s \text{ and } \int_t^T V_s \overleftarrow{dB_s}.
$$

Now we decompose Ψ^n as follows

$$
\Psi_s^n = (\Psi_s^n)^+ - (\Psi_s^n)^-,
$$

where x^+ (resp. x^-) denotes the positive (resp. negative) part of x. Again from Inequality (2.3.29) we deduce that

$$
(1-\varepsilon)\frac{\|Z_s^n\|^2}{(Y_s^n)^{q+2}} - K\frac{1}{(Y_s^n)^q} \le \Psi_s^n = (\Psi_s^n)^+ - (\Psi_s^n)^- \le \frac{\|Z_s^n\|^2}{(Y_s^n)^{q+2}},
$$

and therefore

$$0 \leq (\Psi^n_s)^- \leq K\frac{1}{(Y^n_s)^q} \leq K(qT + \frac{1}{\alpha^q}).$$

Therefore for any $t \in [0,T]$:

$$0 \leq \frac{q(q+1)}{2}\int_t^T (\Psi^n_s)^- ds \leq \frac{q(q+1)}{2}K(qT + \frac{1}{\alpha^q})(T-t). \tag{2.3.30}$$

If we define

$$\Gamma_t = \liminf_{n\to+\infty} \frac{q(q+1)}{2}\mathbb{E}^{\mathcal{G}_t}\int_t^T (\Psi^n_s)^- ds,$$

we obtain that Γ is a non negative bounded process. Since $(\Psi^n)^-$ is non negative, it is straightforward that Γ is a supermartingale. Moreover the dominated convergence theorem proves that Γ is a continuous process such that: $\lim_{t\to T} \Gamma_t = 0$.

Coming back to(2.3.27) and taking the conditional expectation, we have

$$\begin{aligned}\frac{q(q+1)}{2}\mathbb{E}^{\mathcal{G}_t}\int_t^T \Psi^n_s ds &= \mathbb{E}^{\mathcal{G}_t}\left(\frac{1}{(\xi\wedge n)^q}\right) - \frac{1}{(Y^n_t)^q} + q(T-t) \\ &\quad + q\mathbb{E}^{\mathcal{G}_t}\int_t^T \frac{g(s,Y^n_s,Z^n_s)}{(Y^n_s)^{1+q}}\overleftarrow{dB_s},\end{aligned}$$

and the right hand side converges weakly in L^2. Therefore if we define

$$\Theta_t = \limsup_{n\to+\infty}\frac{q(q+1)}{2}\mathbb{E}^{\mathcal{G}_t}\int_t^T (\Psi^n_s)^+ ds,$$

taking the weak limit, we obtain

$$\Theta_t - \Gamma_t = \mathbb{E}^{\mathcal{G}_t}\left(\frac{1}{\xi^q}\right) - \frac{1}{(Y_t)^q} + q(T-t) + q\mathbb{E}^{\mathcal{G}_t}\int_t^T V_s\overleftarrow{dB_s}.$$

We can remark that $(\Theta_t)_{0\leq t<T}$ is a non-negative supermartingale and for any $t \in [0,T]$:

$$\frac{1}{(Y_t)^q} = q(T-t) + \mathbb{E}^{\mathcal{G}_t}\left(\frac{1}{\xi^q}\right) - \Theta_t + \Gamma_t + q\mathbb{E}^{\mathcal{G}_t}\int_t^T V_s\overleftarrow{dB_s}.$$

Θ being a right-continuous non-negative supermartingale, the limit of Θ_t as t goes to T exists $\mathbb{P}$-a.s. and this limit Θ_{T^-} is finite $\mathbb{P}$-a.s. The same holds for the backward Itô integral with limit $M_T = 0$. The L^1-bounded martingale $\mathbb{E}^{\mathcal{G}_t}\left(\frac{1}{\xi^q}\right)$ converges a.s. to $1/\xi^q$, as t goes to T, then the limit of Y_t as $t \to T$ exists and is equal to:

$$\lim_{t\to T,\ t<T} Y_t = \frac{1}{(\frac{1}{\xi^q} - \Theta_{T^-})^{1/q}}.$$

If we were able to prove that Θ_{T^-} is zero a.s., we would have shown that $Y_T = \xi$.

2.3.2.2 The case ξ non negative

Now we just assume that $\xi \geq 0$. We cannot apply the Itô formula to $1/(Y^n)^q$ because we have no positive lower bound for Y^n. We define for $n \geq 1$ and $m \geq 1$, $\xi^{n,m}$ by:

$$\xi^{n,m} = (\xi \wedge n) \vee \frac{1}{m},$$

This random variable is in L^2 and is greater or equal to $1/m$ a.s. , with $\xi^{n,m}$ as terminal condition, has a unique solution $(\widetilde{Y}^{n,m}, \widetilde{Z}^{n,m})$ of our BSDE (2.0.4). Let us come back to (2.3.25). We have already proved that

$$\mathbb{E}\left|\widetilde{Y}_t^{n,m} - Y_t^n\right|^2 \leq e^{(1+K)T}\mathbb{E}\left|\xi^{n,m} - \xi^n\right|^2 \leq e^{(1+K)T}\frac{1}{m^2}.$$

Now using (2.3.24) with $t = 0$ and taking the expectation we obtain first:

$$(1-\varepsilon)\mathbb{E}\int_0^T \left[\widetilde{Z}_r^{n,m} - Z_r^n\right]^2 dr \leq \left(1 + (1+K)Te^{(1+K)T}\right)\frac{1}{m^2}$$

from which we deduce that the two stochastic integrals in (2.3.24) are true martingales. Therefore we can use Burckholder-Davis-Gundy inequality and once again from (2.3.24) there exists a constant C such that:

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left|\widetilde{Y}_t^{n,m} - Y_t^n\right|^2\right) \leq \frac{C}{m^2}.$$

From Fatou's lemma the same inequality holds for $\widetilde{Y}^m - Y$. Since $\widetilde{Y}^m$ has a limit on the left at T, so does Y.

2.3.3 Minimal solution

In this section we will achieve the proof of Theorem 14. Let $(\widetilde{Y}, \widetilde{Z})$ be another non negative solution of BDSDE (2.0.4) in the sense of Definition 8. Note that we will only use that

$$\liminf_{t\to T} \widetilde{Y}_t \geq \xi$$

(and not the stronger condition (D3)). Then a.s. for any $t \in [0,T]$, $Y_t \leq \widetilde{Y}_t$.

Lemma 2. *With the assumptions of Theorem 14, we prove:*

$$\forall t \in [0,T],\ \widetilde{Y}_t \leq \left(\frac{1}{q(T-t)}\right)^{\frac{1}{q}}.$$

Proof. For every $0 < h < T$, we define on $[0, T-h]$

$$\Lambda_h(t) = \left(\frac{1}{q(T-h-t)}\right)^{\frac{1}{q}}.$$

Λ_h is the solution of the ordinary differential equation:

$$\Lambda_h'(t) = (\Lambda_h(t))^{1+q}$$

with final condition $\Lambda_h(T-h) = +\infty$. But on the interval $[0, T-h]$, $(\widetilde{Y}, \widetilde{Z})$ is a solution of the BDSDE (2.0.4) with final condition $\widetilde{Y}_{T-h}$. From the assumptions $\widetilde{Y}_{T-h}$ is in $L^2(\Omega)$, so is finite a.s. Now we take the difference between $\widetilde{Y}$ and Λ_h for all $0 \leq t \leq s < T-h$:

$$\begin{aligned}\hat{Y}_t = \widetilde{Y}_t - \Lambda_h(t) &= \widetilde{Y}_s - \Lambda_h(s) - \int_t^s \left[\left(\widetilde{Y}_r\right)^{1+q} - \Lambda_h(r)^{1+q}\right] dr \\ &+ \int_t^s \left[g(r, \widetilde{Y}_r, \widetilde{Z}_r) - g(r, \Lambda_h(r), 0)\right] \overleftarrow{dB_r} - \int_t^s \widetilde{Z}_r dW_r.\end{aligned}$$

Recall that $g(t, y, 0) = 0$. We apply Itô's formula to $(\hat{Y}_t^+)^2$ between t and s:

$$\begin{aligned}(\hat{Y}_t^+)^2 &\leq (\hat{Y}_s^+)^2 - 2\int_t^s \hat{Y}_r^+ \left(\Lambda_h(r)^{1+q} - \left(\widetilde{Y}_r\right)^{1+q}\right) dr \\ &+ 2\int_t^s \hat{Y}_r^+ g(r, \widetilde{Y}_r, \widetilde{Z}_r)\overleftarrow{dB_r} - 2\int_t^s \hat{Y}_r^+ \widetilde{Z}_r dW_r \\ &+ \int_t^s \mathbf{1}_{\hat{Y}_r > 0} |g(r, \widetilde{Y}_r, \widetilde{Z}_r) - g(r, \Lambda_h(r), 0)|^2 dr - \int_t^s \mathbf{1}_{\hat{Y}_r > 0} |\widetilde{Z}_r|^2 dr\end{aligned}$$

The generator f of this BDSDE satisfies Condition (A1) with $\mu = 0$, and g satisfies (A4). Thus

$$\begin{aligned}(\hat{Y}_t^+)^2 &\leq (\hat{Y}_s^+)^2 + K_g \int_t^s (\hat{Y}_r^+)^2 dr - (1-\varepsilon)\int_t^s \mathbf{1}_{\hat{Y}_r > 0}|\widetilde{Z}_r|^2 dr \\ &+ 2\int_t^s \hat{Y}_r^+ g(r, \widetilde{Y}_r, \widetilde{Z}_r)\overleftarrow{dB_r} - 2\int_t^s \hat{Y}_r^+ \widetilde{Z}_r dW_r.\end{aligned}$$

We take the expectation of both sides. Since $(\widetilde{Y}, \widetilde{Z})$ is in $\mathcal{E}^2(0, s)$, the martingale part disappears and we deduce that:

$$\mathbb{E}(\hat{Y}_t^+)^2 \leq \mathbb{E}(\hat{Y}_s^+)^2 + K_g \int_t^s \mathbb{E}(\hat{Y}_r^+)^2 dr.$$

By Gronwall's inequality we obtain:

$$\mathbb{E}(\hat{Y}_t^+)^2 \leq e^{K_g(s-t)}\mathbb{E}(\hat{Y}_s^+)^2.$$

Remark that for any $0 \leq t \leq T-h$

$$0 \leq \hat{Y}_t^+ \leq \sup_{0 \leq t \leq T-h} \widetilde{Y}_t = \Xi_{T-h}.$$

Since $\widetilde{Y} \in \mathbb{S}^2(0, T-h; \mathbb{R}_+)$, $\Xi_{T-h} \in L^2(\Omega)$. By dominated convergence theorem as s goes to $T-h$:

$$\mathbb{E}(\hat{Y}_t^+)^2 \leq e^{K_g(T-h-t)}\mathbb{E}(\hat{Y}_{T-h}^+)^2 = 0.$$

Thus $\widetilde{Y}_t \leq \Lambda_h(t)$ for all $t \in [0, T-h]$ and for all $0 < h < T$. So it is clear that for every $t \in [0, T]$:

$$\widetilde{Y}_t \leq \left(\frac{1}{q(T-t)}\right)^{\frac{1}{q}}.$$

This achieves the proof of the Lemma. □

Let us prove now minimality of our solution. We will prove that $\widetilde{Y}$ is greater than Y^n for all $n \in \mathbb{N}$, which implies that Y is the minimal solution. Let (Y^n, Z^n) be the solution of the BSDE

(2.0.4) with $\xi \wedge n$ as terminal condition. By comparison with the solution of the same BSDE with the deterministic terminal data n:

$$Y_t^n \leq \left(\frac{1}{q(T-t)+1/n^q}\right)^{1/q} \leq n.$$

Between the instants $0 \leq t \leq s < T$:

$$\begin{aligned}\hat{Y}_t = Y_t^n - \widetilde{Y}_t &= \left(Y_s^n - \widetilde{Y}_s\right) - \int_t^s \left((\widetilde{Y}_r)^{1+q} - (Y_r^n)^{1+q}\right) dr - \int_t^s \left(Z_r^n - \widetilde{Z}_r\right) dW_r \\ &+ \int_t^s \left[g(r, Y_r^n, Z_r^n) - g(r, \widetilde{Y}_r, \widetilde{Z}_r)\right] \overleftarrow{dB_r}.\end{aligned}$$

Once again we apply Itô's formula to $(\hat{Y}_t^+)^2$:

$$\begin{aligned}(\hat{Y}_t^+)^2 &\leq (\hat{Y}_s^+)^2 - 2\int_t^s \hat{Y}_r^+ \left((Y_r^n)^{1+q} - (\widetilde{Y}_r)^{1+q}\right) dr \\ &+ 2\int_t^s \hat{Y}_r^+ \left[g(r, Y_r^n, Z_r^n) - g(r, \widetilde{Y}_r, \widetilde{Z}_r)\right] \overleftarrow{dB_r} - 2\int_t^s \hat{Y}_r^+ \hat{Z}_r dW_r \\ &+ \int_t^s \mathbf{1}_{\hat{Y}_r > 0} |g(r, Y_r^n, Z_r^n) - g(r, \widetilde{Y}_r, \widetilde{Z}_r)|^2 dr - \int_t^s \mathbf{1}_{\hat{Y}_r > 0} |\hat{Z}_r|^2 dr\end{aligned}$$

and we deduce that

$$\mathbb{E}(\hat{Y}_t^+)^2 \leq e^{K_g(s-t)} \mathbb{E}(\hat{Y}_s^+)^2.$$

Since $0 \leq \hat{Y}_t^+ \leq Y_t^n \leq n$, by dominated convergence theorem, we can take the limit as s goes to T and we obtain that $\mathbb{E}(\hat{Y}_t^+)^2 = 0$.

2.4 Limit at time T by localization technic

From now, the process Y is continuous on $[0, T[$ and we define $Y_T = \xi$. The main difficulty will be to prove the continuity at time T. We have already proved that

$$\liminf_{t \to T} Y_t = \lim_{t \to T} Y_t$$

and remarked that Y is lower semi-continuous on $[0, T]$:

$$\xi \leq \liminf_{t \to T} Y_t.$$

In this paragraph we prove that the inequality in (2.3.26) is in fact an equality, i.e.

$$\xi = \liminf_{t \to T} Y_t.$$

Note that the only remaining problem is on the set $\mathcal{R} = \{\xi < +\infty\}$.
From now on, the conditions of Theorem 15 hold. In particular the terminal condition ξ is equal to $h(X_T)$, where h is a function defined on $\mathbb{R}^d$ with values in $\overline{\mathbb{R}^+}$. We denote by $\mathcal{S} = \{h = +\infty\}$ the closed set of singularity and $\mathcal{R}$ its complement. In order to prove continuity at time T, we

will show that for any function θ of class $C^2(\mathbb{R}^d;\mathbb{R}^+)$ with a compact support strictly included in $\mathcal{R}=\{h<+\infty\}$ and for any $t\in[0,T]$, we have

$$\begin{aligned}\mathbb{E}(\xi\theta(X_T)) &= \mathbb{E}(Y_t\theta(X_t))+\mathbb{E}\int_t^T\theta(X_r)(Y_r)^{1+q}dr+\mathbb{E}\int_t^T Y_r\mathcal{L}\theta(X_r)dr \\ &+ \mathbb{E}\int_t^T Z_r.\nabla\theta(X_r)\sigma(r,X_r)dr\end{aligned} \tag{2.4.31}$$

with suitable integrability conditions on the last three terms in the right-hand side. Here $\mathcal{L}$ is the operator:

$$\mathcal{L}=\frac{1}{2}\sum_{i,j}(\sigma\sigma^*)_{ij}(t,x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_i b_i(t,x)\frac{\partial}{\partial x_i}=\frac{1}{2}\mathrm{Trace}\left(\sigma\sigma^*(t,x)D^2\right)+b(t,x).\nabla; \tag{2.4.32}$$

where in the rest of the chapter, ∇ and D^2 will always denote respectively the gradient and the Hessian matrix w.r.t. the space variable. If we let t go to T in Equality (2.4.31) and if we apply Fatou's lemma, we have:

$$\mathbb{E}\left[\xi\theta(X_T)\right]=\lim_{t\to T}\mathbb{E}\left[Y_t\theta(X_t)\right]\geq\mathbb{E}\left[\left(\liminf_{t\to T}Y_t\right)\theta(X_T)\right]. \tag{2.4.33}$$

Note that we need the suitable estimates on the last three terms in (2.4.31). Now recall that we already know (2.3.26). Hence, the inequality in (2.4.33) is in fact a equality, i.e.

$$\mathbb{E}\left[h(X_T)\theta(X_T)\right]=\mathbb{E}\left[\theta(X_T)\left(\liminf_{t\to T}Y_t\right)\right].$$

And with (2.3.26) once again, we conclude that:

$$\lim_{t\to T}Y_t=\liminf_{t\to T}Y_t=h(X_T),\quad \mathbb{P}-\text{a.s. on }\{h(X_T)<\infty\}.$$

In the next subsections we prove that (2.4.31) holds. As in [83] the proof depends on the value of q and we distinguish $q>2$ where no other assumption is needed (the non linearity is "strong enough") and $q\leq 2$ where we have to add additional conditions. Moreover the arguments are almost the same as in [83], thus technical details will be skip here.
Let φ be a function in the class $C^2\left(\mathbb{R}^d\right)$ with a compact support. Let (Y,Z) be the solution of the BDSDE (2.0.4) with the final condition $\zeta\in L^2(\Omega)$. For any $t\in[0,T]$:

$$\begin{aligned}Y_t\varphi(X_t) &= Y_0\varphi(X_0)+\int_0^t\varphi(X_r)\left[Y_r|Y_r|^q dr-g(r,Y_r,Z_r)\overleftarrow{dB_r}+Z_r.dW_r\right] \\ &+\int_0^t Y_r d(\varphi(X_r))+\int_0^t Z_r.\nabla\varphi(X_r)\sigma(r,X_r)dr \\ &= Y_0\varphi(X_0)+\int_0^t\varphi(X_r)Y_r|Y_r|^q dr+\int_0^t Z_r.\nabla\varphi(X_r)\sigma(r,X_r)dr \\ &+\int_0^t Y_r\mathcal{L}\varphi(X_r)dr+\int_0^t\left(Y_r\nabla\varphi(X_r)\sigma(r,X_r)+\varphi(X_r)Z_r\right).dW_r \\ &-\int_0^t\varphi(X_r)g(r,Y_r,Z_r)\overleftarrow{dB_r}\end{aligned}$$

where $\mathcal{L}$ is the operator defined by (2.4.32). Taking the expectation:

$$\begin{aligned}\mathbb{E}(Y_t\varphi(X_t)) &= \mathbb{E}(Y_0\varphi(X_0)) + \mathbb{E}\int_0^t \varphi(X_r)Y_r|Y_r|^q dr \\ &+\mathbb{E}\int_0^t Z_r.\nabla\varphi(X_r)\sigma(r,X_r)dr + \mathbb{E}\int_0^t Y_r\mathcal{L}\varphi(X_r)dr.\end{aligned} \tag{2.4.34}$$

Note that the generator g does not appear in this expression. Hence no extra assumption will be added in the case $q > 2$.

Let U be a bounded open set with a regular boundary and such that the compact set $\overline{U}$ is included in $\mathcal{R}$. We denote by $\Phi = \Phi_U$ a function which is supposed to belong to $C^2(\mathbb{R}^d;\mathbb{R}_+)$ and such that Φ is equal to zero on $\mathbb{R}^d \setminus U$, is positive on U. Let α be a real number such that

$$\alpha > 2(1 + 1/q).$$

For $n \in \mathbb{N}$, let (Y^n, Z^n) be the solution of the BSDE (2.0.4) with the final condition $(h\wedge n)(X_T)$. The equality (2.4.34) with $t = T$ becomes:

$$\begin{aligned}\mathbb{E}(Y_T^n\Phi^\alpha(X_T)) &= \mathbb{E}(Y_0^n\Phi^\alpha(X_0)) + \mathbb{E}\int_0^T Z_r^n.\nabla(\Phi^\alpha)(X_r)\sigma(r,X_r)dr \\ &+ \mathbb{E}\int_0^T \Phi^\alpha(X_r)(Y_r^n)^{1+q}dr + \mathbb{E}\int_0^T Y_r^n\mathcal{L}(\Phi^\alpha)(X_r)dr.\end{aligned} \tag{2.4.35}$$

Lemma 3. *Let p be such that*

$$\frac{1}{p} + \frac{1}{1+q} = 1.$$

Then

$$\Phi^{-\alpha(p-1)}\,|\mathcal{L}(\Phi^\alpha)|^p \in L^\infty([0,T]\times\mathbb{R}^d).$$

Lemma 3 and Hölder inequality show that there exists a constant C such that

$$\forall n\in\mathbb{N},\quad \mathbb{E}\int_0^T |Y_r^n\mathcal{L}(\Phi^\alpha)(X_r)|\,dr \le C\left[\mathbb{E}\int_0^T \Phi^\alpha(X_r)(Y_r^n)^{1+q}dr\right]^{1/(1+q)}. \tag{2.4.36}$$

We distinguish the case $q > 2$ and $q \le 2$ in order to control the term containing Z in (2.4.35).

2.4.1 Proof of (2.4.31) if $q > 2$

Proposition 5 and the Cauchy-Schwarz inequality prove immediatly the following result.

Lemma 4 (Case $q > 2$)**.** *If $q > 2$, then there exists a constant $C = C(q,\Phi,\alpha,\sigma)$ such that for all $n \in \mathbb{N}$:*

$$\mathbb{E}\int_0^T |Z_r^n.\nabla(\Phi^\alpha)(X_r)\sigma(r,X_r)|\,dr \le C. \tag{2.4.37}$$

Lemmata 3 and 4 and Equality (2.4.35) imply the next result.

Lemma 5. *The sequence $\Phi^\alpha(X)(Y^n)^{1+q}$ is a bounded sequence in $L^1(\Omega\times[0,T])$ and with the Fatou lemma, $Y^{1+q}\Phi^\alpha(X)$ belongs to $L^1(\Omega\times[0,T])$:*

$$\mathbb{E}\int_0^T Y_r^{1+q}\Phi^\alpha(X_r)dr < +\infty.$$

Proof. In (2.4.35), since the support of Φ is in F_∞^c and from the previous lemma, the first three terms are bounded w.r.t. n by some constant C. If the sequence is not bounded, from inequality (2.4.36), there is a contradiction. $\square$

Now we prove Equality (2.4.31). Let θ be a function of class $C^2(\mathbb{R}^d;\mathbb{R}^+)$ with a compact support strictly included in $\mathcal{R} = \{h < +\infty\}$. There exists a open set U s.t. the support of θ is included in U and $\overline{U} \subset \mathcal{R}$. Let $\Phi = \Phi_U$ be the previously used function. Let us recall that α is strictly greater than $2(1+1/q) > 2$. Thanks to a result in the proof of the lemma 2.2 of [64], there exists a constant $C = C(\theta, \alpha)$ such that:

$$|\theta| \leq C\Phi^\alpha \ , \ |\nabla\theta| \leq C\Phi^{\alpha-1} \text{ and } \left\|D^2\theta\right\| \leq C\Phi^{\alpha-2}.$$

Using Lemma 5 and the monotone convergence theorem, we have

$$\lim_{n\to+\infty} \mathbb{E}\int_t^T (Y_r^n)^{1+q}\theta(X_r)dr = \mathbb{E}\int_t^T (Y_r)^{1+q}\theta(X_r)dr,$$

with

$$\mathbb{E}\int_0^T Y_r^{1+q}\theta(X_r)dr \leq C. \tag{2.4.38}$$

We can do the same calculations using the previously given estimations on θ, $\nabla\theta$ and $D^2\theta$ in terms of power of Φ^α and Hölder's inequality:

$$\Phi^{-\alpha(p-1)}\left|\mathcal{L}\theta\right|^p \in L^\infty([0,T]\times\mathbb{R}^d).$$

Now we can write:

$$Y_r^n\mathcal{L}\theta(X_r) = \left(Y_r^n\Phi^{\alpha/(1+q)}\right)\left(\Phi^{-\alpha/(1+q)}\mathcal{L}\theta(X_r)\right) = \left(Y_r^n\Phi^{\alpha/(1+q)}\right)\left(\Phi^{-\alpha(p-1)/p}\mathcal{L}\theta(X_r)\right).$$

The sequence $Y^n\Phi^{\alpha/(1+q)} = Y^n\Phi^{\alpha(1-1/p)}$ is a bounded sequence in $L^{1+q}(\Omega\times[0,T])$ (see Lemma 5). Therefore using a weak convergence result and extracting a subsequence if necessary, we can pass to the limit in the term:

$$\mathbb{E}\int_t^T Y_r^n\mathcal{L}\theta(X_r)dr,$$

with

$$\mathbb{E}\int_0^T \left|Y_r\mathcal{L}\theta(X_r)\right|dr \leq C. \tag{2.4.39}$$

Recall the estimation (2.3.23): there exists a constant C such that for all $n\in\mathbb{N}$:

$$\mathbb{E}\int_0^T \|Z_r^n\|^2(T-r)^{\frac{2}{q}}dr \leq C.$$

Hence, there exists a subsequence, which we still denote $Z^n(T-r)^{1/q}$, and which converges weakly in the space $L^2\left(\Omega\times(0,T), d\mathbb{P}\times dt;\mathbb{R}^d\right)$ to a limit, and the limit is $Z(T-r)^{1/q}$, because we already know that Z^n converges to Z in $L^2\left(\Omega\times(0,T-\delta)\right)$ for all $\delta>0$. $\nabla\theta(X)\sigma(.,X)(T-.)^{-1/q}$ is $L^2\left(\Omega\times(0,T)\right)$, because θ is compactly supported and $q>2$. Therefore,

$$\lim_{n\to+\infty}\mathbb{E}\int_t^T Z_r^n.\nabla\theta(X_r)\sigma(r,X_r)dr = \mathbb{E}\int_t^T Z_r.\nabla\theta(X_r)\sigma(r,X_r)dr.$$

And Lemma 4 shows that

$$\mathbb{E}\int_0^T \left|Z_r.\nabla\theta(X_r)\sigma(r,X_r)\right|dr \leq C. \tag{2.4.40}$$

To conclude we write Equality (2.4.34) for (Y^n,Z^n) and θ and we pass to the limit. This gives Equality (2.4.31) with the three estimates (2.4.38), (2.4.39) and (2.4.40).

2.4.2 Proof of (2.4.31) if $q \leq 2$

If we just assume $q > 0$, Lemma 4 does not hold anymore. In other words our previous control on the term containing Z in (2.4.31) fails. But if we are able to prove that there exists a function ψ such that for $0 < t \leq T$:

$$\mathbb{E}\int_t^T Z_r^n.\nabla\theta(X_r)\sigma(r,X_r)dr = \mathbb{E}\int_t^T Y_r^n\psi(r,X_r)dr, \tag{2.4.41}$$

then we apply again the Hölder inequality in order to control

$$\mathbb{E}\int_t^T Y_r^n\psi(r,X_r)dr \quad \text{by} \quad \mathbb{E}\int_t^T (Y_r^n)^{1+q}\Phi^\alpha(X_r)dr.$$

Note that we will add the following condition on g. From now on g is a measurable function defined on $[0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^k$ such that the Lipschitz property (A4) holds and for any $(t,x,x',y,z)\in[0,T]\times(\mathbb{R}^d)^2\times\mathbb{R}\times\mathbb{R}^m$

$$|g(t,x,y,z)-g(t,x',y,z)| \leq K_g|x-x'|. \tag{A6}$$

First we will prove the next result. In fact it is a quite straightforward modification of Proposition 2.3 in [74] or Proposition 4.2 in [5]. Let us denote by $\mathcal{B}^2(0,T;\mathbb{D}^{1,2})$ the set of processes (Y,Z) such that $Y\in\mathbb{S}^2([0,T])$, $Z\in\mathbb{H}^2(0,T)$ and Y_t and Z_t belong to $\mathbb{D}^{1,2}$ with

$$\mathbb{E}\left[\int_0^T(|Y_t|^2+\|Z_t\|^2)dt+\int_0^T\int_0^T(|D_sY_t|^2+\|D_sZ_t\|^2)dtds\right] < +\infty.$$

Lemma 6. *Assume that (Y,Z) is solution of the BSDE (2.0.4) with terminal condition $\xi = h(X_T)$:*

$$Y_t = h(X_T) - \int_t^T Y_s|Y_s|^q ds + \int_t^T g(s,X_s,Y_s,Z_s)\overleftarrow{dB_s} - \int_t^T Z_s dW_s$$

where h is a bounded Lipschitz function on $\mathbb{R}^d$, g satisfies the previous Lipschitz condition and $X_s\in\mathbb{D}^{1,2}$ for every $s\in[0,T]$.

1. *Then $(Y,Z)\in\mathcal{B}^2(0,T;\mathbb{D}^{1,2})$ and for all $1\leq i\leq d$, $\left\{D_s^iY_s, 0\leq s\leq T\right\}$ is a version of $\{(Z_s)_i, 0\leq s\leq T\}$. $(Z_s)_i$ denotes the i-th component of Z_s. Here, $D_s^iY_s$ has the following sense:*

$$D_s^iY_s = \lim_{\substack{r\to s\\ r<s}} D_r^iY_s.$$

2. *There exist two random fields u and v such that:*

$$Y_t = u(t,X_t) \quad \text{and} \quad Z_t = v(t,X_t)$$

and for any $(t,x)\in[0,T]\times\mathbb{R}^m$, $u(t,x)$ and $v(t,x)$ are $\mathcal{F}_{t,T}^B$-measurable.

Proof. We just sketch the proof. The technical arguments can be found in [5] or [36]. It is known (see e.g. proof of Theorem 1.1 in [74]) that the solution (Y,Z) of the previous BSDE is obtained by passing to the limit in the following iteration scheme:

$$(Y^0,Z^0) = (0,0)$$

$$Y_t^{m+1} = h(X_T) - \int_t^T Y_s^m |Y_s^m|^q ds + \int_t^T g(s, X_s, Y_s^m, Z_s^m) \overleftarrow{dB_s} - \int_t^T Z_s^{m+1} dW_s.$$

Our goal here is to show by induction that for all $m \in \mathbb{N}$ $(Y^m, Z^m) \in \mathcal{B}^2(0,T;\mathbb{D}^{1,2})$ and that (Y^m, Z^m) converges in $L^2([0,T], \mathbb{D}^{1,2} \times \mathbb{D}^{1,2})$ to (Y, Z).

It is clear that this is true at step 0 since constants are Malliavin differentiable. Let us suppose now that $(Y^m, Z^m) \in \mathcal{B}^2(0,T;\mathbb{D}^{1,2})$. By Proposition 1.2.4 in [67] we know that $h(X_T) \in \mathbb{D}^{1,2}$ and $g(s, X_s, Y_s^m, Z_s^m) \in \mathbb{D}^{1,2}$, since h and g are supposed Lipschitz continuous and X_T, Y_s^m and Z_s^m are in $\mathbb{D}^{1,2}$. With Lemma 4.2 in [5] we obtain that $\int_t^T g(s, X_s, Y_s^m, Z_s^m) d\overleftarrow{B}_s \in \mathbb{D}^{1,2}$. Moreover, since h is bounded, Y^m is also bounded (see (2.3.18)) and $x \mapsto x|x|^q \in C^1(\mathbb{R})$. Therefore $Y_t^m |Y_t^m|^q \in \mathbb{D}^{1,2}$. Then following the arguments of section 7.1 in [5], we obtain that $(Y^{m+1}, Z^{m+1}) \in \mathcal{B}^2(0,T;\mathbb{D}^{1,2})$ with $D_s Y_t^{m+1} = D_s Z_t^{m+1} = 0$ for $0 \leq t \leq s \leq T$ and for $0 \leq s \leq t \leq T$

$$\begin{aligned} D_s Y_t^{m+1} &= H_T^x (D_s X_T) - (q+1) \int_t^T |Y_u^m|^q D_s Y_u^m du \\ &+ \int_t^T \left[G_u^{x,m} D_s X_u + G_u^{y,m} D_s Y_u^m + G_u^{z,m} D_s Z_u^m \right] d\overleftarrow{B}_u - \int_t^T D_s Z_u^{m+1} dW_u \end{aligned}$$

where H_T^x, resp. $G_u^{x,m}$, $G_u^{y,m}$ and $G_u^{z,m}$ are four bounded random variables with H^x (resp. G_u^x, G_u^y and G_u^z) is $\mathcal{F}_T^W$ (resp. $\mathcal{F}_u$) measurable. The bound depends only on the Lipschitz constant of h and g. Using the same arguments as in [5] we can prove that $G^{x,m}$, $G^{y,m}$ and $G^{z,m}$ converge to bounded processes and (Y^m, Z^m) converges in $L^2([0,T], \mathbb{D}^{1,2} \times \mathbb{D}^{1,2})$ to (Y, Z). Now for $0 < s < t$

$$\begin{aligned} D_s Y_t &= Z_s + (q+1) \int_s^t |Y_u|^q D_s Y_u du \\ &- \int_s^t \left[G_u^x D_s X_u + G_u^y D_s Y_u + G_u^z D_s Z_u \right] d\overleftarrow{B}_u + \int_s^t D_s Z_u dW_u. \end{aligned}$$

We then pass to the limit as s goes to t to obtain the desired result.

For the second part, we will show that there exists two random fields u^m and v^m, such that for any $(t,x) \in [0,T] \times \mathbb{R}^m$, $u^m(t,x)$ and $v^m(t,x)$ are $\mathcal{F}_{t,T}^B$-measurable and

$$Y_t^m = u^m(t, X_t) \quad \text{and} \quad Z_t^m = v^m(t, X_t).$$

It is clear that this is true at step 0 by taking $u^0 = v^0 = 0$. Let us suppose now that there exists two functions satisfying the measurability as stated in the proposition such that at step m our processes verify:

$$Y_t^m = u^m(t, X_t) \quad \text{and} \quad Z_t^m = v^m(t, X_t).$$

We can write then:

$$\begin{aligned} Y_t^{m+1} &= h(X_T) - \int_t^T u^m(s, X_s) |u^m(s, X_s)|^q ds \\ &+ \int_t^T g(s, X_s, u^m(s, X_s), v^m(s, X_s)) d\overleftarrow{B}_s - \int_t^T Z_s^{m+1} dW_s \end{aligned}$$

We take the expectation with respect to $\mathcal{G}_t$, which leads to:

$$\begin{aligned} Y_t^{m+1} &= \mathbb{E}^{\mathcal{G}_t} \Bigg[h(X_T) - \int_t^T u^m(s, X_s) |u^m(s, X_s)|^q ds \\ &\qquad + \int_t^T g(s, X_s, u^m(s, X_s), v^m(s, X_s)) d\overleftarrow{B}_s \Bigg]. \end{aligned}$$

By Markov property, there exists a function u^{m+1} with $u^{m+1}(t,x)$ is $\mathcal{F}_T^B$-measurable and

$$Y_t^{m+1} = u^{m+1}(t, X_t).$$

We know that Y_t^{m+1} is $\mathcal{F}_t$-measurable and we deduce that $u^{m+1}(t,x)$ is in fact $\mathcal{F}_{t,T}^B$-measurable. The convergence of u^m and v^m can be obtained by the same arguments as in [36], section 4. □
The last assumption implies that $x \mapsto h(x) \wedge n$ is a bounded Lipschitz function on $\mathbb{R}^d$. Moreover these conditions imply that $X_s \in \mathbb{D}^{1,2}$. Therefore we can apply the previous proposition to (Y^n, Z^n) to establish:

Proposition 6. *There exists a function ψ such that for every $s \in [0,T]$*

$$\mathbb{E}\left[Z_s^n \nabla\theta(X_s)\sigma(s,X_s)\right] = \mathbb{E}\left[Y_s^n \psi(s,X_s)\right],$$

where ψ is given by:

$$\begin{aligned} \psi(t,x) &= -\sum_{i=1}^{d} (\nabla\theta(x)\sigma(t,x))_i \frac{div(p\sigma^i)(t,x)}{p(t,x)} \\ &- Trace(D^2\theta(x)\sigma\sigma^*(t,x)) - \sum_{i=1}^{d} \nabla\theta(x).\nabla\sigma^i(t,x)\sigma^i(t,x). \end{aligned} \tag{2.4.42}$$

Proof. We would like to apply here the integration by parts formula (Lemma 1.2.2 in [67]) which states:

$$\mathbb{E}[G\langle DF, h\rangle_H] = \mathbb{E}[-F\langle DG, h\rangle_H + FGW(h)],$$

where $H := \mathbb{L}^2([0,T],\mathbb{R}^d)$ and F and G are two random variables in $\mathbb{D}^{1,2}$. But in order to use this we need to rewrite our expectation to make appear the scalar product in H. Actually we have:

$$\mathbb{E}\left[Z_t^n.\nabla\theta(X_t)\sigma(t,X_t)\right] = \mathbb{E}\left[D_t Y_t^n.\nabla\theta(X_t)\sigma(t,X_t)\right] = \sum_{i=1}^{d} \mathbb{E}\left[D_t^i Y_t^n (\nabla\theta\sigma)_i(X_t)\right]$$

where $(\nabla\theta\sigma)(X_t) = \nabla\theta(X_t)\sigma(t,X_t)$ and $(\nabla\theta\sigma)_i(X_t)$ denotes the i-th component of $(\nabla\theta\sigma)(X_t)$. As in the proof of Proposition 17 in [83] we can use the following approximation:

$$\mathbb{P} - a.s. \quad D_t^i Y_t^n = \lim_{j\to\infty} \langle DY_t^n, \nu_j^i\rangle_H = \lim_{j\to\infty} \int_0^T j\mathbf{1}_{[t-\frac{1}{j},t[}(s)(D_s Y_t^n).e_i ds$$

$(e_i)_{i=1,\dots,d}$ being the canonical basis of $\mathbb{R}^d$.
Let $t \in [0,T]$, by integrations by parts formula we get:

$$\begin{aligned} \mathbb{E}[\langle DY_t^n, \nu_j^i\rangle_H (\nabla\theta\sigma)_i(X_t)] &= \mathbb{E}\left[Y_t^n (\nabla\theta\sigma)_i(X_t) \int_0^T \nu_j^i(s) dW_s\right] \\ &- \mathbb{E}\left[Y_t^n \int_0^T \nu_j^i(s) D_s((\nabla\theta\sigma)_i(X_t)) ds\right]. \end{aligned}$$

Let us consider the first term:

$$\begin{aligned}
\mathbb{E}\left[Y_t^n(\nabla\theta\sigma)_i(X_t)\int_0^T \nu_j^i(s)dW_s\right] &= j\mathbb{E}\left[Y_t^n(\nabla\theta\sigma)_i(X_t)\left(W_t^i - W_{t-\frac{1}{j}}^i\right)\right]\\
&= j\mathbb{E}\left[u^n(t,X_t)(\nabla\theta\sigma)_i(X_t)\left(W_t^i - W_{t-\frac{1}{j}}^i\right)\right]\\
&= j\mathbb{E}\left[\mathbb{E}\left[u^n(t,X_t)\Big|\mathcal{F}_t^W\right](\nabla\theta\sigma)_i(X_t)\left(W_t^i - W_{t-\frac{1}{j}}^i\right)\right]\\
&= j\mathbb{E}\left[v^n(t,X_t)(\nabla\theta\sigma)_i(X_t)\left(W_t^i - W_{t-\frac{1}{j}}^i\right)\right]
\end{aligned}$$

since $u^n(t,x)$ is $\mathcal{F}_{t,T}^B$-measurable and B and W are independent. Now

$$\begin{aligned}
&\mathbb{E}\left[v^n(t,X_t)(\nabla\theta\sigma)_i(X_t)\left(W_t^i - W_{t-\frac{1}{j}}^i\right)\right]\\
&\quad = -\mathbb{E}\left[v^n(t,X_t)(\nabla\theta\sigma)_i(X_t)\int_{t-\frac{1}{j}}^t \frac{\operatorname{div}(\sigma^i p)(u,X_u)}{p(u,X_u)}du\right]
\end{aligned}$$

where p is the density of X and σ^i is the i-th column of the matrix σ. As in [83], we use Lemmas 3.1 and 4.1 in [71] with the same arguments. For convenience let us just recall that from Theorems 7 and 10 in [4], Theorem II.3.8 of [92] and Theorem III.12.1 in [58], the density $p(x;.,.)$ exists and satisfies:

- $p(x;.,.) \in L^2(\delta,T;H^2)$ for all $\delta > 0$;
- p is Hölder continuous in x and satisfies the following inequality for $s \in]0,T]$:

$$\frac{\exp\left(-C\frac{|y-x|^2}{s}\right)}{Cs^{m/2}} \le p(x;s,y) \le \frac{C\exp\left(-\frac{|y-x|^2}{Cs}\right)}{s^{m/2}}; \tag{2.4.43}$$

- $y \mapsto \partial p/\partial y_i(x;.,.)$ is Hölder continuous in y.

In this proof we omit the variable x in $p(x;.,.)$. The previous properties of p ensure that $\operatorname{div}(p\sigma^i)/p$ is well-defined and regular. Let j go to $+\infty$, we have:

$$\begin{aligned}
\mathbb{E}\left[D_t^i Y_t^n(\nabla\theta\sigma)_i(X_t)\right] &= -\mathbb{E}\left[Y_t^n(\nabla\theta\sigma)_i(X_t)\frac{\operatorname{div}(\sigma^i p)(t,X_t)}{p(t,X_t)}\right]\\
&- \mathbb{E}\left[Y_t^n D_t(\nabla\theta(X_t)\sigma(t,X_t))\right].
\end{aligned}$$

From the regularity assumptions on θ and σ we can compute directly the last term to obtain:

$$\mathbb{E}\left[Z_t^n.\nabla\theta(X_t)\sigma(t,X_t)\right] = \mathbb{E}\left[Y_t^n\psi(t,X_t)\right]$$

with ψ given by (2.4.42). □

As in the case $q > 2$, we have to prove that Equality (2.4.31) holds with suitable integrability conditions. Now Equation (2.4.35) becomes:

$$\begin{aligned}
\mathbb{E}(Y_T^n\Phi^\alpha(X_T)) &= \mathbb{E}(Y_0^n\Phi^\alpha(X_0)) + \mathbb{E}\int_0^T Z_r^n.\nabla(\Phi^\alpha)(X_r)\sigma(r,X_r)dr\\
&+\mathbb{E}\int_0^T \Phi^\alpha(X_r)(Y_r^n)^{1+q}dr + \mathbb{E}\int_0^T Y_r^n\mathcal{L}(\Phi^\alpha)(X_r)dr\\
&= \mathbb{E}(Y_0^n\Phi^\alpha(X_0)) + \mathbb{E}\int_0^T \Phi^\alpha(X_r)(Y_r^n)^{1+q}dr + \mathbb{E}\int_0^T Y_r^n\Psi_\alpha(r,X_r)dr
\end{aligned} \tag{2.4.44}$$

with Ψ_α the following function: for $t \in]0,T]$ and $x \in \mathbb{R}^d$

$$\begin{aligned}\Psi_\alpha(t,x) &= \nabla(\Phi^\alpha)(x).b(t,x) - \frac{1}{2}\text{Trace}(D^2(\Phi^\alpha)(x)\sigma\sigma^*(t,x)) \\ &- \sum_{i=1}^d \left((\nabla(\Phi^\alpha)(x)\sigma(t,x))_i \; \frac{\text{div}(p(t,x)\sigma^i(t,x))}{p(t,x)} \right) \\ &- \sum_{i=1}^d \left(\nabla(\Phi^\alpha)(x).[\nabla\sigma^i(t,x)\sigma^i(t,x)] \right).\end{aligned}$$

In [83] it is proved that for a fixed $\varepsilon > 0$ and $p = 1 + 1/q$:

$$\Phi^{-\alpha(p-1)} |\Psi_\alpha|^p \in L^\infty([\varepsilon, T] \times \mathbb{R}^d).$$

If it is true, then the last term in (2.4.45) satisfies:

$$\mathbb{E}\int_t^T |Y_r^n \Psi_\alpha(r, X_r)|\, dr \leq C \left(\mathbb{E}\int_t^T \Phi^\alpha(X_r)(Y_r^n)^{1+q} dr \right)^{\frac{1}{1+q}}$$

and the end of the proof will be the same as in the case $q > 2$.

2.5 Link with SPDE's

In the introduction, we have said that there is a connection between doubly stochastic backward SDE whose terminal data is a function of the value at time T of a solution of a SDE (or forward-backward system), and solutions of a large class of semilinear parabolic stochastic PDE. Let us precise this connection in our case.

To begin with, we modify the equation (2.1.8). For all $(t,x) \in [0,T] \times \mathbb{R}^d$, we denote by $X^{t,x}$ the solution of the following SDE:

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dW_r, \text{ for } s \in [t,T], \tag{2.5.45}$$

and $X_s^{t,x} = x$ for $s \in [0,t]$. We enforce the assumptions on b and σ: we assume that b (resp. σ) is a C^2 (resp. C^3) function whose partial derivatives of order less than 2 (resp. 3) are bounded. Therefore b and σ satisfy the assumptions (L)-(G). We consider the following doubly stochastic BSDE for $t \leq s \leq T$:

$$Y_s^{t,x} = h(X_T^{t,x}) - \int_s^T Y_r^{t,x}|Y_r^{t,x}|^q dr + \int_s^T g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\overleftarrow{dB_r} - \int_s^T Z_r^{t,x} dW_r, \tag{2.5.46}$$

where h is a function defined on $\mathbb{R}^d$ with values in $\mathbb{R}$. The two equations (2.5.45) and (2.5.46) are called a forward-backward system. This system is connected with the stochastic PDE (2.0.5) with terminal condition h.

More precisely for any $n \in \mathbb{N}^*$, let $(Y^{n,t,x}, Z^{n,t,x})$ be the solution of the BDSDE (2.5.46) with terminal condition $h(X_T^{t,x}) \wedge n$. We know that

$$0 \leq Y_s^{n,t,x} \leq \left(\frac{1}{q(T-s) + \frac{1}{n^q}} \right)^{\frac{1}{q}} \leq n.$$

And the generator $y \mapsto -y|y|^q$ is Lipschitz continuous on the interval $[-n, n]$. Moreover if we assume Assumption (H3), then $h \wedge n$ is a Lipschitz and bounded function on $\mathbb{R}^d$. Hence $h \wedge n$ belongs to $L^2(\mathbb{R}^d, \rho(x)dx) = L^2_\rho(\mathbb{R}^d)$. Recall that ρ is defined by (2.1.9) and here we just need that $\kappa > d$. Since we have imposed that $g(t, x, y, 0) = 0$, now the all conditions in [6] are satisfied.

Proposition 7 (Theorem 3.1 in [6])**.** *There exists a unique weak solution $u^n \in \mathcal{H}(0, T)$ of the SPDE* (2.0.5) *with terminal function $h \wedge n$. Moreover $u^n(t, x) = Y_t^{n,t,x}$ and*

$$Y_s^{n,t,x} = u^n(s, X_s^{t,x}), \quad Z_s^{n,t,x} = (\sigma^* \nabla u^n)(s, X_s^{t,x}).$$

The space $\mathcal{H}(0, T)$ is defined in Section 2.1. The random field u^n is a weak solution of (2.0.5) , which means:

1. For some $\delta > 0$
$$\sup_{s \leq T} \mathbb{E}\left[||u^n(s,.)||^{1+\delta}_{L^2_\rho(\mathbb{R}^d)} \right] < \infty. \tag{WS1}$$
Since u^n is bounded by n a.s. it is true for any $\delta > 0$.

2. For every test-function $\phi \in C^\infty(\mathbb{R}^d)$, $dt \otimes d\mathbb{P}$ a.e.
$$\lim_{s \uparrow t} \int_{\mathbb{R}^d} u^n(s, x)\phi(x)dx = \int_{\mathbb{R}^d} u^n(t, x)\phi(x)dx. \tag{WS2}$$

3. Finally u^n satisfies for every function $\Psi \in C_c^{1,\infty}([0, T] \times \mathbb{R}^d)$

$$\begin{aligned} \int_t^T \int_{\mathbb{R}^d} u^n(s, x)\partial_s \Psi(s, x)dxds + \int_{\mathbb{R}^d} u^n(t, x)\Psi(t, x)dx - \int_{\mathbb{R}^d} (h(x) \wedge n)\Psi(T, x)dx \\ -\frac{1}{2} \int_t^T \int_{\mathbb{R}^d} (\sigma^* \nabla u^n)(s, x)(\sigma^* \nabla \Psi)(s, x)dxds \\ - \int_t^T \int_{\mathbb{R}^d} u^n(s, x)\mathrm{div}\left(\left(b - \widetilde{A} \right) \Psi \right)(s, x)dxds \\ = -\int_t^T \int_{\mathbb{R}^d} \Psi(s, x)u^n(s, x)|u^n(s, x)|^q dxds \\ + \int_t^T \int_{\mathbb{R}^d} \Psi(s, x)g(s, x, u^n(s, x), \sigma^* \nabla u^n(s, x))dx\overleftarrow{dB_s}. \end{aligned} \tag{2.5.47}$$

$C_c^{1,\infty}([0, T] \times \mathbb{R}^d)$ is the set of functions $\psi : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ such that ψ has a compact support w.r.t. $x \in \mathbb{R}^d$ and
$$\widetilde{A}_i = \frac{1}{2} \sum_{k=1}^d \frac{\partial(\sigma\sigma^*)_{k,i}}{\partial x_k}.$$

Remember that we have defined a process $(Y^{t,x}, Z^{t,x})$ solution in the sense of the Definition 8 of the backward doubly stochastic differential equation (2.5.46) with singular terminal condition h (see Theorem 14 and the beginning of the section 2.4 on continuity at time T). The process Y is obtained as the increasing limit of the processes Y^n:

$$Y_s^{t,x} = \lim_{n \to +\infty} Y_s^{n,t,x} \quad \text{a.s.}.$$

Therefore we can define the following random field u as follows:

$$u(t,x) = Y_t^{t,x} = \lim_{n\to+\infty} Y_t^{n,t,x} = \lim_{n\to+\infty} u^n(t,x).$$

Our aim is to prove Theorem 16, that is, u is also a weak solution of (2.0.5) with the singular terminal condition h. For any n we have a.s.

$$0 \le Y_s^{n,t,x} \le \left(\frac{1}{q(T-s)+\frac{1}{n^q}}\right)^{\frac{1}{q}} \le \left(\frac{1}{q(T-s)}\right)^{\frac{1}{q}}.$$

In particular for any (t,x)

$$0 \le u^n(t,x) \le \left(\frac{1}{q(T-t)}\right)^{\frac{1}{q}}$$

and hence u satisfies the same estimate. Thus u is bounded on $[0,t]\times\mathbb{R}^d$ and in $L^2_\rho(\mathbb{R}^d)$. By dominated convergence theorem, for any $\delta > 0$, u satisfies (WS1) and (WS2) for any $0 \le s \le t \le T-\delta$.

Moreover we have $Z_s^{n,t,x} = (\sigma^*\nabla u^n)(s, X_s^{t,x})$ and from the proof on Theorem 14 we know that the sequence of processes $(Z_s^{n,t,x},\ s \ge t)$ converges in $L^2((0,T-\delta)\times\Omega)$ for any $\delta > 0$ to $Z^{t,x}$. Hence the sequence u^n converges in $\mathcal{H}(0,t)$ to u. From Proposition 5 we have the a priori estimate (2.3.23):

$$\mathbb{E}\int_0^T (T-s)^{2/q}\|Z_s^{n,t,x}\|^2 ds \le \frac{8+KT}{1-\varepsilon}\left(\frac{1}{q}\right)^{2/q}$$

(as usual $Z_s^{n,t,x} = 0$ if $s < t$). Therefore we deduce

$$\mathbb{E}\int_0^T (T-s)^{2/q}\|(\sigma^*\nabla u^n)(s, X_s^{t,x})\|^2 ds \le \frac{8+KT}{1-\varepsilon}\left(\frac{1}{q}\right)^{2/q}.$$

We multiply each side by $\rho(x)$, we integrate w.r.t. x and we use Proposition 5.1 in [6] to have:

$$\mathbb{E}\int_{\mathbb{R}^d}\int_0^T (T-s)^{2/q}\|(\sigma^*\nabla u^n)(s,x)\|^2\rho(x)dxds \le C$$

where the constant C does not depend on n. With the Fatou lemma we have the same inequality for u.

$$\mathbb{E}\int_{\mathbb{R}^d}\int_0^t (T-s)^{2/q}\|(\sigma^*\nabla u)(s,x)\|^2\rho(x)dxds < \infty.$$

Now for every function $\Psi \in C_c^{1,\infty}([0,T]\times\mathbb{R}^d)$, u^n satisfies (2.5.47), therefore for every $0 \le r \le t < T$, u^n satisfies also:

$$\begin{aligned}
&\int_r^t\int_{\mathbb{R}^d} u^n(s,x)\partial_s\Psi(s,x)dxds + \int_{\mathbb{R}^d} u^n(r,x)\Psi(r,x)dx - \int_{\mathbb{R}^d} u^n(t,x)\Psi(t,x)dx \\
&\quad -\frac{1}{2}\int_r^t\int_{\mathbb{R}^d}(\sigma^*\nabla u^n)(s,x)(\sigma^*\nabla\Psi)(s,x)dxds \\
&\quad -\int_r^t\int_{\mathbb{R}^d} u^n(s,x)\mathrm{div}\left(\left(b-\widetilde{A}\right)\Psi\right)(s,x)dxds \\
&= -\int_r^t\int_{\mathbb{R}^d}\Psi(s,x)u^n(s,x)|u^n(s,x)|^q dxds \\
&\quad +\int_r^t\int_{\mathbb{R}^d}\Psi(s,x)g(s,x,u^n(s,x),\sigma^*\nabla u^n(s,x))dx\overleftarrow{dB_s}.
\end{aligned} \tag{2.5.48}$$

But using monotone convergence theorem or the convergence of u^n to u in $\mathcal{H}(0,t)$, we can pass to the limit as n goes to $+\infty$ in (2.5.48) and we obtain that u is a weak solution of (2.0.5) on $[0,T-\delta]\times\mathbb{R}^d$ for any $\delta>0$.
The only trouble concerns the behavior of u near T. Since u^n satisfies (2.5.47) for any n, and by the integrability or regularity assumptions on u^n, b and σ, we have

$$\lim_{t\to T}\int_{\mathbb{R}^d} u^n(t,x)\psi(x)dx = \int_{\mathbb{R}^d}(h(x)\wedge n)\psi(x)dx$$

for any function $\psi\in C_c^\infty(\mathbb{R}^d)$. Therefore by monotonicity

$$\liminf_{t\uparrow T}\int_{\mathbb{R}^d} u(t,x)\psi(x)dx \geq \liminf_{t\uparrow T}\int_{\mathbb{R}^d} u^n(t,x)\psi(x)dx = \int_{\mathbb{R}^d}(h(x)\wedge n)\psi(x)dx$$

for any n. Hence a.s.

$$\liminf_{t\uparrow T}\int_{\mathbb{R}^d} u(t,x)\psi(x)dx \geq \int_{\mathbb{R}^d} h(x)\psi(x)dx.$$

Our aim is to prove the converse inequality with the lim sup. In the second section we have proved Equation (2.4.31) with suitable integrability condition on all terms:

$$\begin{aligned}\mathbb{E}(h(X_T^{t,x})\theta(X_T^{t,x})) &= \mathbb{E}(u(t,x)\theta(x)) + \mathbb{E}\int_t^T \theta(X_r^{t,x})(Y_r^{t,x})^{1+q}dr \\ &+ \mathbb{E}\int_t^T Y_r^{t,x}\mathcal{L}\theta(X_r^{t,x})dr + \mathbb{E}\int_t^T Z_r^{t,x}.\nabla\theta(X_r^{t,x})\sigma(r,X_r^{t,x})dr\end{aligned}$$

for any smooth functions θ such that its compact support is strictly included in $\mathcal{R}=\{h<+\infty\}$. If we integrate this w.r.t. dx (no weight function ρ is needed here since θ is of compact support) and we let t go to T we obtain:

$$\lim_{t\to T}\int_{\mathbb{R}^d}\mathbb{E}(h(X_T^{t,x})\theta(X_T^{t,x}))dx = \lim_{t\to T}\int_{\mathbb{R}^d}\mathbb{E}(u(t,x)\theta(x))dx.$$

By dominated convergence theorem this gives:

$$\lim_{t\to T}\mathbb{E}\left(\int_{\mathbb{R}^d} u(t,x)\theta(x)dx\right) = \int_{\mathbb{R}^d} h(x)\theta(x)dx$$

for any function $\theta\in C_c^2(\mathbb{R}^d)$ with $\text{supp}(\theta)\cap\mathcal{S}=\emptyset$.
Note that Fatou's lemma implies that

$$\mathbb{E}\left(\liminf_{t\to T}\int_{\mathbb{R}^d} u(t,x)\theta(x)dx\right) \leq \int_{\mathbb{R}^d} h(x)\theta(x)dx$$

and thus we obtain that a.e. on $\Omega\times\mathbb{R}^d$

$$\liminf_{t\to T} u(t,x) = h(x).$$

Remark 2. *If g does not depend of Z (or on ∇u), and if $g\in C_b^{0,2,3}([0,T]\times\mathbb{R}^d\times R;\mathbb{R}^d)$, then from [21], u^n is a stochastically bounded viscosity solution of the SPDE (2.0.5) on $[0,T]\times\mathbb{R}^d$ and u is also a stochastically bounded viscosity solution of the SPDE (2.0.5) on $[0,T-\delta]\times\mathbb{R}^d$ for any $\delta>0$.*

Now let $\widetilde{u}$ be a non negative weak solution of (2.0.5) on $[0, T - \delta] \times \mathbb{R}^d$ for any $\delta > 0$. It means that for any $\delta > 0$, $\widetilde{u} \in \mathcal{H}(0, T - \delta)$ and $\widetilde{u}$ satisfies (WS1) and (WS2) and (2.5.48) on $[0, T - \delta]$. Moreover we assume that $\liminf_{t \to T} \widetilde{u}(t, x) \geq h(x)$ a.e. on $\Omega \times \mathbb{R}^d$. We follow the proof of uniqueness for Theorem 3.1 in [6]. We define

$$\widetilde{Y}_s^{t,\cdot} = \widetilde{u}(s, X_s^{t,\cdot}), \quad \widetilde{Z}_s^{t,\cdot} = \widetilde{(\sigma^* \nabla u)}(s, X_s^{t,\cdot}).$$

Then by the same arguments as in [6], $(\widetilde{Y}_s^{t,x}, \widetilde{Z}_s^{t,x})$ solves the BDSDE (2.0.4) on any interval $[0, T - \delta]$. Moreover a.s.

$$\liminf_{s \to T} \widetilde{Y}_s^{t,x} \geq h(X_T^{t,x}) = \xi.$$

By Lemma 2 and the proof of minimality of the solution of the BDSDE (Theorem 14), we deduce that for every n, a.s.

$$Y_s^{t,x,n} \leq \widetilde{Y}_s^{t,x} \leq \left(\frac{1}{q(T - s)} \right)^{1/q}.$$

Thus $u(t, x) \leq \widetilde{u}(t, x)$ a.e. on $\Omega \times [0, T] \times \mathbb{R}^d$. And u is the minimal weak solution of the SPDE (2.0.5) with singular terminal condition.

CHAPTER 3

BSDE with jumps and PIDE with singular terminal condition

Contents

3.1 Introduction

In this chapter we would like to provide existence results for the following BSDE:

$$Y_t = \xi - \int_t^T Y_s|Y_s|^q ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e)\tilde{\mu}(ds, de) \tag{3.1.1}$$

where the terminal condition ξ satisfies

$$\mathbb{P}(\xi = +\infty) > 0. \tag{3.1.2}$$

The first aim is the following. We want to find sufficient conditions to have a.s.

$$\lim_{t \to T} Y_t = \xi. \tag{3.1.3}$$

This problem was studied in [83] when there is no jump. In the Markovian framework and for $q > 2$, Equality (3.1.3) was proved. Here we will follow the same idea. But we need other technical assumptions on the jumps of the solution X of the forward SDE and the set $\mathcal{S} = \{\xi = +\infty\}$. The second part is devoted to the study of the related partial integro differential equation (PIDE in short): for any $x \in \mathbb{R}^d$, $u(T, x) = g(x)$ and for any $(t, x) \in [0, T[\times\mathbb{R}^d$

$$\frac{\partial}{\partial t}u(t, x) + \mathcal{L}u(t, x) + \mathcal{I}(t, x, u) - u(t, x)|u(t, x)|^q = 0 \tag{3.1.4}$$

where $\mathcal{L}$ is given above and $\mathcal{I}$ is a integro-differential operator:

$$\mathcal{I}(t,x,\phi) = \int_E [\phi(x+h(t,x,e)) - \phi(x) - (\nabla\phi)(x)h(t,x,e)]\lambda(de).$$

The chapter is organized as follows. In the first part we give the mathematical framework and we recall the construction of the minimal solution of the BSDE (3.1.1). We complete the results of Kruse and Popier [53] with an a priori estimate of the coefficients Z and U. In the second section, we prove Equality (3.1.3). More precisely we prove that the limit always exists and in the Markovian framework and for $q > 2$, this limit is a.s. equal to ξ. Some technical assumptions **(C)** are assumed: they concern the jumps of the forward process X and the set of singularity $\mathcal{S}$. Note that we cannot extend the result to the case $q \leq 2$. In [83] for $q \leq 2$, Malliavin calculus is used to obtain the right estimate. Even if the solution of the BSDE with jumps has Malliavin derivatives (see [28]), we are not able to do the suitable integration by parts and this case is left for further developments.

In the last section we consider the PIDE (3.1.4) and we show that the minimal solution Y of the BSDE provides the minimal viscosity solution u for the PIDE. We show that $Y_t^{t,x} = u(t,x)$ is a (discontinuous) viscosity solution of (3.1.4) on any interval $[0, T-\varepsilon]$ for $\varepsilon > 0$ and under assumptions **(C)**, $u(t,x)$ converges to $g(x)$ as t goes to T. Then we prove minimality of this solution, which requires a comparison result for viscosity solution for PIDE adapted for our setting.

3.2 Setting, construction of the minimal solution

Our setting is the same as in [9]. In the following we will consider $W = (W_t)_{t\in\mathbb{R}_+}$ the standard Brownian Motion on $\mathbb{R}^k$, μ a Poisson random measure on $\mathbb{R}_+ \times E$ with compensator $dt\lambda(de)$. Here $E := \mathbb{R}^\ell \setminus \{0\}$, $\mathcal{E}$ its Borel field and we assume that we have a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$ and the filtration $(\mathcal{F}_t,\ t \geq 0)$ is generated by the two independent processes W and μ and we assume that $\mathcal{F}_0$ contains all $\mathbb{P}$-null elements of $\mathcal{F}$. We will denote $\tilde{\mu}$ is the compensated measure: for any $A \in \mathcal{E}$ such that $\lambda(A) < +\infty$, then $\tilde{\mu}([0,t]\times A) = \mu([0,t]\times A) - t\lambda(A)$ is a martingale. λ is assumed to be a σ-finite measure on $(E, \mathcal{E})$ satisfying

$$\int_E (1 \wedge |e|^2)\lambda(de) < +\infty.$$

In this chapter for a given $T \geq 0$, we denote:

- $\mathcal{P}$: the predictable σ-field on $\Omega \times [0,T]$ and
$$\widetilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{E}$$
- On $\widetilde{\Omega} = \Omega \times [0,T] \times E$, a function that is $\widetilde{\mathcal{P}}$-measurable, is called predictable. $G_{loc}(\mu)$ is the set of $\widetilde{\mathcal{P}}$-measurable functions ψ on $\widetilde{\Omega}$ such that for any $t \geq 0$ a.s.
$$\int_0^t \int_E (|\psi_s(e)|^2 \wedge |\psi_s(e)|)\lambda(de) < +\infty.$$
- $\mathcal{D}$ (resp. $\mathcal{D}(0,T)$): the set of all progressively measurable càdlàg processes on $\mathbb{R}_+$ (resp. on $[0,T]$).

We refer to [49] (see also [12]) for details on random measures and stochastic integrals. Now to define the solution of our BSDE, let us introduce the following spaces for $p \geq 1$.

- $\mathbb{S}^p(0,T)$ is the space of all processes $X \in \mathcal{D}(0,T)$ such that

$$\mathbb{E}\left(\sup_{t\in[0,T]} |X_t|^p \right) < +\infty.$$

  For simplicity, $X_* = \sup_{t\in[0,T]} |X_t|$.

- $\mathbb{H}^p(0,T)$ is the subspace of all processes $X \in \mathcal{D}(0,T)$ such that

$$\mathbb{E}\left[\left(\int_0^T |X_t|^2 dt \right)^{p/2} \right] < +\infty.$$

- $\mathbb{L}^p_\mu(0,T) = L^p_\mu(\Omega \times (0,T) \times E)$: the set of processes $\psi \in G_{loc}(\mu)$ such that

$$\mathbb{E}\left[\left(\int_0^T \int_E |\psi_s(e)|^2 \lambda(de) ds \right)^{p/2} \right] < +\infty.$$

- $L^p_\lambda = L^p(E, \lambda; \mathbb{R}^d)$: the set of measurable functions $\psi : E \to \mathbb{R}^d$ such that

$$\|\psi\|^p_{L^p_\lambda} = \int_E |\psi(e)|^p \lambda(de) < +\infty.$$

- $\mathbb{B}^p_\mu(0,T) = \mathbb{S}^p(0,T) \times \mathbb{H}^p(0,T) \times \mathbb{L}^p_\mu(0,T)$.

In [53], the following result is proved.

Theorem 17 (and Definition)**.** *Let $q > 0$ and ξ a $\mathcal{F}_t$-measurable non negative random variable such that $\mathbb{P}(\xi = \infty) > 0$. There exists a process (Y, Z, U) such that*

- *(Y, Z, U) belongs to $\mathbb{S}^2(0,t)$ for any $t < T$.*
- *for all $0 \leq s \leq t < T$:*

$$Y_s = Y_t - \int_s^t Y_r |Y_r|^q dr - \int_s^t Z_r dW_r - \int_s^t \int_E U_s(e) \tilde{\mu}(ds, de).$$

- *(Y, Z, U) is a super-solution in the sense that: a.s.*

$$\liminf_{t\to T} Y_t \geq \xi.$$

*Any process $(\tilde{Y}, \tilde{Z}, \tilde{U})$ satisfying the previous three items is called **super-solution** of the BSDE (3.1.1) with singular terminal condition ξ.*

By construction, $Y_t \geq 0$ a.s. and it is the minimal one: if $(\tilde{Y}, \tilde{Z}, \tilde{U})$ is another non negative super-solution, then a.s. $\tilde{Y}_t \geq Y_t$.
In this section we sketch the construction of the minimal solution and we give an a priori estimate of the martingale part (Z, U) of the solution. Indeed we have already seen that if ξ belongs to $\mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$ we have existence and uniqueness results for such BSDE's according to [28] or [52]. The approach in [53] is then to approximate our BSDE by considering a terminal condition of the form $\xi^n := \xi \wedge n$ and observe asymptotic behaviour.
Let us consider the following process:

$$Y_t^n = \xi \wedge n - \int_t^T Y_s^n |Y_s^n|^q ds - \int_t^T Z_s^n dW_s - \int_t^T \int_E U_s^n(e) \tilde{\mu}(ds, de)$$

where (Y^n, Z^n, U^n) is solution of the BSDE. Moreover using comparison argument (see [85] or [52]) we can obtain for $n \leq m$:

$$0 \leq Y_t^n \leq Y_t^m \leq \left(\frac{1}{q(T-t) + \frac{1}{m^q}} \right)^{1/q} \leq \left(\frac{1}{q(T-t)} \right)^{1/q}$$

This allows us to define Y as the increasing limit of the sequence $(Y_t^n)_{n \geq 1}$:

$$\forall\, t \in [0, T], \quad Y_t := \lim_{n \to \infty} Y_t^n.$$

The next lemma shows that the sequences of processes Z^n and U^n also converge.

Lemma 7. *There exists a constant C such that for any m, n and $0 \leq t \leq s < T$:*

$$\mathbb{E}\left[\sup_{0 \leq t \leq s} |Y_t^n - Y_t^m|^2 + \int_0^s \|Z_r^n - Z_r^m\|^2 dr + \int_0^s \int_E |U_r^n(e) - U_r^m(e)|^2 \lambda(de) dr \right] \tag{3.2.5}$$
$$\leq C \mathbb{E}\left[|Y_s^n - Y_s^m|^2 \right].$$

Proof. From the Itô formula with $\phi(y) = y^2$ we deduce for $t \leq s < T$:

$$|Y_t^n - Y_t^m|^2 + \int_t^s \|Z_r^n - Z_r^m\|^2 dr + \int_t^s \int_E |U_r^n(e) - U_r^m(e)|^2 \lambda(de) dr =$$
$$|Y_s^n - Y_s^m|^2 + 2 \int_t^s |Y_r^n - Y_r^m| [(Y_r^n)^{q+1} - (Y_r^m)^{q+1}] dr$$
$$-2 \int_t^s (Y_r^n - Y_r^m)(Z_r^n - Z_r^m) dW_r - \int_t^s \int_E |Y_r^n - Y_r^m|^2 - |Y_{r^-}^n - Y_{r^-}^m|^2 \tilde{\mu}(dr, de).$$

Then from the monotonicity of $y \mapsto y^{q+1}$,

$$|Y_t^n - Y_t^m|^2 + \int_t^s \|Z_r^n - Z_r^m\|^2 dr + \int_t^s \int_E |U_r^n(e) - U_r^m(e)|^2 \lambda(de) dr \tag{3.2.6}$$
$$\leq |Y_s^n - Y_s^m|^2 - 2 \int_t^s (Y_r^n - Y_r^m)(Z_r^n - Z_r^m) dW_r$$
$$- \int_t^s \int_E \left(|Y_r^n - Y_r^m|^2 - |Y_{r^-}^n - Y_{r^-}^m|^2 \right) \tilde{\mu}(dr, de)$$

From (3.2.6) we get directly:

$$\mathbb{E}\left[\int_0^s \|Z_r^n - Z_r^m\|^2 dr + \int_0^s \int_E |U_r^n(e) - U_r^m(e)|^2 \lambda(de) dr \right] \leq \mathbb{E}\left[|Y_s^n - Y_s^m|^2 \right] \tag{3.2.7}$$

because $(Y_r^n - Y_r^m) \in \mathbb{L}^\infty$, $(Z_r^n - Z_r^m) \in \mathbb{L}^2$ and $(U_r^n - U_r^m) \in \mathbb{L}^2$. Thus all stochastic integrals in (3.2.6) are true martingales. Note that (3.2.6) can be also written as follows:

$$\begin{aligned}
&|Y_t^n - Y_t^m|^2 + \int_t^s \|Z_r^n - Z_r^m\|^2 dr + \int_t^s \int_E |U_r^n(e) - U_r^m(e)|^2 \mu(de, dr) \\
&\leq |Y_s^n - Y_s^m|^2 - 2\int_t^s (Y_r^n - Y_r^m)(Z_r^n - Z_r^m) dW_r \\
&-2\int_t^s \int_E (U_r^n(e) - U_r^m(e))(Y_{r^-}^n - Y_{r^-}^m)\tilde{\mu}(dr, de)
\end{aligned}$$

Taking the supremum on $[0, s]$ and using Burkholder-Davis-Gundy inequality we obtain:

$$\begin{aligned}
\mathbb{E} \sup_{t\in[0,s]} |Y_t^n - Y_t^m|^2 &\leq \mathbb{E}|Y_s^n - Y_s^m|^2 + 2\mathbb{E} \sup_{t\in[0,s]} \left| \int_t^s (Y_r^n - Y_r^m)(Z_r^n - Z_r^m) dW_r \right| \\
&+2\mathbb{E} \sup_{t\in[0,s]} \left| \int_t^s \int_E (U_r^n(e) - U_r^m(e))(Y_{r^-}^n - Y_{r^-}^m)\tilde{\mu}(dr, de) \right| \\
&\leq \mathbb{E}|Y_s^n - Y_s^m|^2 + 8\mathbb{E} \left(\int_0^s (Y_r^n - Y_r^m)^2 (Z_r^n - Z_r^m)^2 dr \right)^{1/2} \\
&+8\mathbb{E} \left(\int_0^s \int_E (Y_{r^-}^n - Y_{r^-}^m)^2 (U_r^n(e) - U_r^m(e))^2 \lambda(de) dr \right)^{1/2}.
\end{aligned}$$

By Young's inequality we have:

$$\begin{aligned}
\mathbb{E} \sup_{t\in[0,s]} |Y_t^n - Y_t^m|^2 &\leq \mathbb{E}|Y_s^n - Y_s^m|^2 + \frac{1}{4}\mathbb{E} \sup_{t\in[0,s]} |Y_t^n - Y_t^m|^2 + 32\mathbb{E} \int_0^s (Z_r^n - Z_r^m)^2 dr \\
&+\frac{1}{4}\mathbb{E} \sup_{t\in[0,s]} |Y_t^n - Y_t^m|^2 + 32\mathbb{E} \int_0^s \int_E (U_r^n(e) - U_r^m(e))^2 \lambda(de) dr.
\end{aligned}$$

Thus we deduce that

$$\begin{aligned}
\mathbb{E} \sup_{t\in[0,s]} |Y_t^n - Y_t^m|^2 &\leq 2\mathbb{E}|Y_s^n - Y_s^m|^2 + 64\mathbb{E} \int_0^s (Z_r^n - Z_r^m)^2 dr \\
&+64\mathbb{E} \int_0^s \int_E (U_r^n(e) - U_r^m(e))^2 \lambda(de) dr.
\end{aligned}$$

Using (3.2.7) we obtain the desired inequality. □

Now we know that for any $s \in [0, T - \varepsilon]$, $Y_s \in \mathbb{L}^\infty$, and since Y_s^n converges to Y_s almost surely, we can deduce from the dominated convergence theorem combined with inequality (3.2.5):

1. For every $\varepsilon > 0$, $(Z_n)_{n\geq 1}$ is a Cauchy sequence in $\mathbb{H}^2(0, T - \varepsilon)$, and converges to $Z \in \mathbb{H}^2(0, T - \varepsilon)$.

2. For every $\varepsilon > 0$, $(U_n)_{n\geq 1}$ is a Cauchy sequence in $\mathbb{L}_\mu^2(0, T - \varepsilon)$, and converges to $U \in \mathbb{L}_\mu^2(0, T - \varepsilon)$.

3. $(Y^n)_{n\geq 1}$ converges to Y in $\mathbb{D}^2(0, T - \varepsilon)$.

4. (Y, Z, U) satisfies for every $0 \leq s < T$, for all $0 \leq t \leq s$

$$Y_t = Y_s - \int_t^s (Y_r)^{1+q} dr - \int_t^s Z_r dB_r - \int_t^s \int_E U_r(e)\tilde{\mu}(dr, de)$$

Recall that on Y we have the following estimate:

$$0 \leq Y_t \leq \left(\frac{1}{q(T-t)}\right)^{1/q}.$$

The previous proof shows that

$$\mathbb{E}\left[\int_0^t \|Z_s\|^2 ds + \int_0^t \int_E |U_s(e)|^2 \lambda(de) ds\right] \leq \left(\frac{1}{q(T-t)}\right)^{2/q}.$$

Now let us prove some sharper estimates on Z and U.

Proposition 8. *The process (Z, U) satisfies:*

$$\mathbb{E}\left[\int_0^T (T-s)^{2/q} \left(\|Z_s\|^2 + \int_E |U_s(e)|^2 \lambda(de)\right) ds\right] \leq 16 \left(\frac{1}{q}\right)^{2/q}.$$

Proof. The proof is almost the same as in [83], Proposition 10. First suppose there exists a constant $\alpha > 0$ such that $\mathbb{P}$-a.s. $\xi \geq \alpha$. In this case, by comparison, for all integer n and all $t \in [0, T]$:

$$Y_t^n \geq \left(\frac{1}{qT + 1/\alpha^q}\right)^{1/q} > 0.$$

Let $\delta > 0$ and $\theta : \mathbb{R} \to \mathbb{R}$, $\theta_q : \mathbb{R} \to \mathbb{R}$ defined by:

$$\left\{\begin{array}{rcll} \theta(x) & = & \sqrt{x} & \text{on } [\delta, +\infty[, \\ \theta(x) & = & 0 & \text{on }]-\infty, 0], \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{rcll} \theta_q(x) & = & x^{\frac{1}{2q}} & \text{on } [\delta, +\infty[, \\ \theta_q(x) & = & 0 & \text{on }]-\infty, 0], \end{array}\right.$$

and such that θ and θ_q are non-negative, non-decreasing and in respectively $C^2(\mathbb{R})$ and $C^1(\mathbb{R})$. We apply the Itô formula on $[0, T-\delta]$ to the function $\theta_q(T-t)\theta(Y_t^n)$, with $\delta < (qT + 1/\alpha^q)^{-1/q}$:

$$\begin{aligned}
\theta_q(\delta)\theta(Y_{T-\delta}^n) - \theta_q(T)\theta(Y_0^n) &= \frac{1}{2q}\int_0^{T-\delta} (T-s)^{1/2q-1}(Y_s^n)^{1/2} ds \\
+\frac{1}{2}\int_0^{T-\delta} (T-s)^{1/2q}(Y_s^n)^{-1/2} dY_s^n &- \frac{1}{8}\int_0^{T-\delta} (T-s)^{1/2q}(Y_s^n)^{3/2}(Z_s^n)^2 ds \\
+ \quad \int_0^{T-\delta} (T-s)^{1/2q} \int_E &\left[(Y_s^n)^{1/2} - (Y_{s_-}^n)^{1/2} - \frac{1}{2}(Y_{s_-}^n)^{-1/2} U_s^n(e)\right] \mu(ds, de)
\end{aligned}$$

We deduce:

$$\begin{aligned}
& \frac{1}{8}\mathbb{E}\left[\int_0^{T-\delta} (T-s)^{1/2q}(Y_s^n)^{-3/2}(Z_s^n)^2 ds\right] \\
- \quad & \mathbb{E}\left[\int_0^{T-\delta} \int_E (T-s)^{1/2q} \left((Y_s^n)^{1/2} - (Y_{s_-}^n)^{1/2} - \frac{1}{2}(Y_{s_-}^n)^{-1/2} U_s^n(e)\right) \mu(ds, de)\right] \\
= \quad & \mathbb{E}\left[\theta_q(T)\theta(Y_0^n) - \theta_q(\delta)\theta(Y_{T-\delta}^n)\right] + \frac{1}{2}\mathbb{E}\left[\int_0^{T-\delta} (T-s)^{1/2q}(Y_s^n)^{1/2} \left((Y_s^n)^q - \frac{1}{q(T-s)}\right) ds\right] \\
\leq \quad & \mathbb{E}\left[\theta_q(T)\theta(Y_0^n)\right] \leq \left(\frac{1}{q}\right)^{1/2q}
\end{aligned}$$

Therefore

$$\mathbb{E}\left[\int_0^{T-\delta}(T-s)^{1/2q}(Y_s^n)^{-3/2}(Z_s^n)^2 ds\right] \leq 8\left(\frac{1}{q}\right)^{1/2q}.$$

We have also

$$\mathbb{E}\left[\int_0^{T-\delta}(T-s)^{1/2q}(Y_s^n)^{-3/2}(Z_s^n)^2 ds\right] \geq q^{3/2q}\mathbb{E}\left[\int_0^{T-\delta}(T-s)^{2/q}(Z_s^n)^2 ds\right].$$

So if we combine those two inequalities we get:

$$\mathbb{E}\left[\int_0^{T-\delta}(T-s)^{2/q}(Z_s^n)^2 ds\right] \leq 8\left(\frac{1}{q}\right)^{2/q}.$$

We can conclude to get the first part of our proposition by letting $\delta \to 0$ and using Fatou's Lemma.
We now want to prove the second part. We already know:

$$-\mathbb{E}\left[\int_0^{T-\delta}\int_E (T-s)^{1/2q}\left((Y_s^n)^{1/2}-(Y_{s_-}^n)^{1/2}-\frac{1}{2}(Y_{s_-}^n)^{-1/2}U_s^n(e)\right)\mu(ds,de)\right] \leq \left(\frac{1}{q}\right)^{1/2q}.$$

Let $Y_s^{n,\alpha} := Y_{s_-}^n + \alpha U_s^n(e)$. We have:

$$(Y_s^n)^{1/2}-(Y_{s_-}^n)^{1/2}-\frac{1}{2}(Y_{s_-}^n)^{-1/2}U_s^n(e) = \int_0^1 |U_s^n(e)|^2(-\frac{1}{4})(Y_s^{n,\alpha})^{-3/2}(1-\alpha)d\alpha.$$

One can easily check that: $Y_s^{n,\alpha} \leq Y_{s_-}^n \vee Y_s^n \leq (\frac{1}{q(T-s)})^{1/q}$. So we can deduce now:

$$\frac{1}{4}\mathbb{E}\left[\int_0^{T-\delta}\int_E\int_0^1 q^{3/2q}(T-s)^{2/q}(1-\alpha)|U_s^n(e)|^2 d\alpha\mu(ds,de)\right] \leq \left(\frac{1}{q}\right)^{1/2q}$$

and so:

$$\mathbb{E}\left[\int_0^{T-\delta}\int_E (T-s)^{2/q}|U_s^n(e)|^2\mu(ds,de)\right] \leq 8\left(\frac{1}{q}\right)^{2/q}.$$

Now we come back to the case $\xi \geq 0$. We can not apply the Itô formula because we do not have any positive lower bound for Y^n. We will approach Y^n in the following way. We define for $n \geq 1$ and $m \geq 1$, $\xi^{n,m}$ by:

$$\xi^{n,m} = (\xi \wedge n) \vee \frac{1}{m}.$$

This random variable is in L^2 and is greater or equal to $1/m$ a.s. The BSDE (3.1.1), with $\xi^{n,m}$ as terminal condition, has a unique solution $(\widetilde{Y}^{n,m}, \widetilde{Z}^{n,m}, \widetilde{U}^{n,m})$. It is immediate that if $m \leq m'$ and $n \leq n'$ then:

$$\widetilde{Y}^{n,m'} \leq \widetilde{Y}^{n',m}.$$

As for the sequence Y^n, we can define $\widetilde{Y}^m$ as the limit when n grows to $+\infty$ of $\widetilde{Y}^{n,m}$. This limit $\widetilde{Y}^m$ is greater than $Y = \lim_{n\to+\infty} Y^n$. But for any m and n, for $t \in [0,T]$:

$$
\begin{aligned}
\left|\widetilde{Y}^{n,m}_t - Y^n_t\right|^2 &= |\xi^{n,m} - \xi^n|^2 - 2\int_t^T \left[\widetilde{Y}^{n,m}_r - Y^n_r\right]\left[\left(\widetilde{Y}^{n,m}_r\right)^{q+1} - (Y^n_r)^{q+1}\right] dr \\
&\quad -2\int_t^T \left[\widetilde{Y}^{n,m}_r - Y^n_r\right]\left[\widetilde{Z}^{n,m}_r - Z^n_r\right] dW_r - \int_t^T \left[\widetilde{Z}^{n,m}_r - Z^n_r\right]^2 dr \\
&\quad - \int_t^T \int_E |\widetilde{U}^{n,m}_r(e) - U^n_r(e)|^2 \lambda(de)dr \\
&\quad - \int_t^T \int_E \left(|\widetilde{Y}^{n,m}_r - Y^n_r|^2 - |\widetilde{Y}^{n,m}_{r^-} - Y^n_{r^-}|^2\right)\tilde{\mu}(dr,de). \\
&\leq |\xi^{n,m} - \xi^n|^2 - 2\int_t^T \left[\widetilde{Y}^{n,m}_r - Y^n_r\right]\left[\widetilde{Z}^{n,m}_r - Z^n_r\right] dW_r \\
&\quad - \int_t^T \int_E \left(|\widetilde{Y}^{n,m}_r - Y^n_r|^2 - |\widetilde{Y}^{n,m}_{r^-} - Y^n_{r^-}|^2\right)\tilde{\mu}(dr,de).
\end{aligned}
$$

and taking the expectation:

$$
\mathbb{E}\left|\widetilde{Y}^{n,m}_t - Y^n_t\right|^2 \leq \mathbb{E}\,|\xi^{n,m} - \xi^n|^2 \leq \frac{1}{m^2}. \tag{3.2.8}
$$

To conclude we fix $\delta > 0$ and we apply the Itô formula to the process $(T-.)^{2/q}\left|\widetilde{Y}^{n,m} - Y^n\right|^2$. This leads to the inequality:

$$
\begin{aligned}
&\mathbb{E}\int_0^{T-\delta} (T-r)^{2/q}\left(\left|\widetilde{Z}^{n,m}_r - Z^n_r\right|^2 + \int_E \left|\widetilde{U}^{n,m}_r(e) - U^n_r(e)\right|^2\right) dr \\
&\leq \frac{2}{q}\mathbb{E}\int_0^{T-\delta} (T-s)^{(2/q)-1}\left|\widetilde{Y}^{n,m}_s - Y^n_s\right|^2 ds + (\delta)^{2/q}\mathbb{E}\left|\widetilde{Y}^{n,m}_{T-\delta} - Y^n_{T-\delta}\right|^2.
\end{aligned}
$$

Let δ go to 0 in the previous inequality. We can do that because $(T-.)^{(2/q)-1}$ is integrable on the interval $[0,T]$ and because of (3.2.8). Finally we have

$$
\begin{aligned}
&\mathbb{E}\int_0^{T-\delta} (T-r)^{2/q}\left(\left|\widetilde{Z}^{n,m}_r - Z^n_r\right|^2 + \int_E \left|\widetilde{U}^{n,m}_r(e) - U^n_r(e)\right|^2\right) dr \\
&\leq \frac{1}{m^2}\left[\frac{2}{q}\int_0^T (T-s)^{(2/q)-1} ds\right] = \frac{T^{2/q}}{m^2}.
\end{aligned}
$$

Therefore, for all $\eta > 0$:

$$
\begin{aligned}
&\mathbb{E}\int_0^{T-\delta} (T-r)^{2/q}\left(|Z^n_r|^2 + \int_E |U^n_r(e)|^2\right) dr \\
&\leq (1+\eta)\mathbb{E}\int_0^T (T-r)^{2/q}\left(\left|\widetilde{Z}^{n,m}_r\right|^2 + \int_E \left|\widetilde{U}^{n,m}_r(e)\right|^2\right) dr \\
&+ \quad (1+\frac{1}{\eta})\mathbb{E}\int_0^T (T-r)^{2/q}\left(\left|\widetilde{Z}^{n,m}_r - Z^n_r\right|^2 + \int_E \left|\widetilde{U}^{n,m}_r(e) - U^n_r(e)\right|^2\right) dr \\
&\leq \quad (1+\eta)16(1/q)^{2/q} + (1+\frac{1}{\eta})\frac{T^{2/q}}{m^2}.
\end{aligned}
$$

We have applied the previous result to $\widetilde{Z}^{n,m}$ and $\widetilde{U}^{n,m}$. Now we let first m go to $+\infty$ and then η go to 0, we have:

$$\mathbb{E}\int_0^T (T-r)^{2/q}\left(|Z_r^n|^2+\int_E |U_r^n(e)|^2\right)dr \leq 16(1/q)^{2/q}.$$

The result follows by letting finally n go to ∞ and this achieves the proof of the proposition. □

Let use recall the minimality of this solution.

Proposition 9 (Minimal solution)**.** *The solution (Y,Z,U) obtained by approximation is minimal. If $(\widetilde{Y},\widetilde{Z},\widetilde{U})$ is another non negative super-solution, then for all $t\in[0,T]$, $\mathbb{P}$-a.s. $\widetilde{Y}_t \geq Y_t$. Moreover*

$$\widetilde{Y}_t \leq \left(\frac{1}{q(T-t)}\right)^{1/q}.$$

Proof. See [53] (or Section 3 in [83]). □

3.3 Behaviour of Y at time T

Let us remark that a.s.

$$\xi\wedge n = \liminf_{t\to T} Y_t^n \leq \liminf_{t\to T} Y_t,$$

and thus we immediately have

$$\xi \leq \liminf_{t\to T} Y_t.$$

Our main problem is to prove that $\xi = \lim_{t\to T} Y_t$. In this section we will prove that the limit always exists and then we will prove that the limit is ξ under some sufficient conditions.

3.3.1 Existence of the limit

We proceed as in [83] (see here for more details). To begin with we suppose that $\xi\geq\alpha>0$. Let (Y^n,Z^n,U^n) be the solution of the BSDE (3.1.1) with terminal condition $\xi\wedge n$, and (Y,Z,U) is the limit of (Y^n,Z^n,U^n). By comparison we have a.s.

$$Y_t \geq Y_t^n \geq \left(\frac{1}{q(T-t)+\frac{1}{\alpha^q}}\right)^{1/q} \geq \left(\frac{1}{qT+\frac{1}{\alpha^q}}\right)^{1/q} > 0.$$

Proposition 10. *The process Y can be written as follows:*

$$Y_t = \left(q(T-t)+\mathbb{E}^{\mathcal{F}_t}\left[\frac{1}{\xi^q}\right]-\phi_t\right)^{-1/q}$$

where ϕ is a non negative supermartingale.

Proof. First, by Itô's formula we obtain:

$$\begin{aligned}(Y_t^n)^{-q} &= (\xi\wedge n)^{-q}+q(T-t)+q\int_t^T \frac{Z_s^n}{(Y_{s_-}^n)^{q+1}}dW_s + q\int_t^T\int_E \frac{U_s^n(e)}{(Y_{s_-}^n)^{q+1}}\tilde{\mu}(de,ds)\\ &- \frac{q(q+1)}{2}\int_t^T \frac{\|Z_s^n\|^2}{(Y_{s_-}^n)^{q+2}}ds - \int_t^T\int_E\left[(Y_s^n)^{-q}-(Y_{s_-}^n)^{-q}+q(Y_{s_-}^n)^{-q-1}U_s^n(e)\right]\mu(ds,de).\end{aligned}$$

Since Y_t^n is bounded from below by some positive constant we can apply Itô's formula. Then taking condition expectation with respect to $\mathcal{F}_t$ we get:

$$\begin{aligned}
(Y_t^n)^{-q} &= \mathbb{E}^{\mathcal{F}_t}[(\xi \wedge n)^{-q}] + q(T-t) - \frac{q(q+1)}{2}\mathbb{E}^{\mathcal{F}_t}\left[\int_t^T \frac{\|Z_s^n\|^2}{(Y_{s_-}^n)^{q+2}} ds\right] \\
&- \mathbb{E}^{\mathcal{F}_t}\left[\int_t^T \int_E \left[(Y_s^n)^{-q} - (Y_{s_-}^n)^{-q} + q(Y_{s_-}^n)^{-q-1} U_s^n(e)\right] \mu(ds, de)\right].
\end{aligned}$$

We will denote from now on:

$$\begin{aligned}
\phi_t^n &= \frac{q(q+1)}{2}\mathbb{E}^{\mathcal{F}_t}\left[\int_t^T \frac{\|Z_s^n\|^2}{(Y_{s_-}^n)^{q+2}} ds\right] \\
&+ \mathbb{E}^{\mathcal{F}_t}\left[\int_t^T \int_E \left[(Y_s^n)^{-q} - (Y_{s_-}^n)^{-q} + q(Y_{s_-}^n)^{-q-1} U_s^n(e)\right] \mu(ds, de)\right].
\end{aligned}$$

We estimate now the following difference for $m \geq n$:

$$0 \leq (Y_t^n)^{-q} - (Y_t^m)^{-q} = \mathbb{E}^{\mathcal{F}_t}\left[(\xi \wedge n)^{-q} - (\xi \wedge m)^{-q}\right] - (\phi_t^n - \phi_t^m).$$

And so we obtain:

$$|\phi_t^n - \phi_t^m| \leq \mathbb{E}^{\mathcal{F}_t}\left[(\xi \wedge n)^{-q} - (\xi \wedge m)^{-q}\right] \vee \left[(Y_t^n)^{-q} - (Y_t^m)^{-q}\right].$$

Since the sequences: $(\mathbb{E}^{\mathcal{F}_t}[(\xi \wedge n)^{-q}])_{n \geq 1}$ and $((Y_t^n)^{-q})_{n \geq 1}$ converge a.s. and in $\mathbb{L}^1$, we deduce $(\phi_t^n)_{n \geq 1}$ converge a.s. and in $\mathbb{L}^1$ to some ϕ_t. So by passing to the limit, one can write:

$$Y_t^{-q} = \mathbb{E}^{\mathcal{F}_t}[\xi^{-q}] + q(T-t) - \phi_t$$

On the other hand it is easy to verify that ϕ is a non negative càdlàg bounded supermartingale:

$$0 \leq \phi_t \leq \mathbb{E}^{\mathcal{F}_t}[\xi^{-q}].$$

This achieves the proof. □

So we can deduce the existence of the following limit:

$$\phi_{T_-} := \lim_{t \nearrow T} \phi_t$$

And so Y_{T_-} exists and is equal to:

$$Y_{T_-} := \lim_{t \nearrow T} Y_t = \left(\frac{1}{\xi^q} - \phi_{T_-}\right)^{-1/q}$$

Now we handle the case were ξ is only supposed to be bounded away from zero. We will consider in that situation the following terminal condition:

$$\xi^{n,m} := (\xi \wedge n) \vee \frac{1}{m}.$$

As before we will denote the associated solution of our BSDE by $(\widetilde{Y}^{n,m}, \widetilde{Z}^{n,m}, \widetilde{U}^{n,m})$. We easily obtain that a.s.

$$0 \leq \widetilde{Y}_t^{n,m} - Y_t^n \leq \frac{1}{m}$$

and thus a.s.

$$\sup_{t\in[0,T]}\left|\widetilde{Y}_t^{n,m}-Y_t^n\right|\le\frac{1}{m}.$$

Let n go to $+\infty$ to deduce:

$$\sup_{t\in[0,T]}\left|\widetilde{Y}_t^{m}-Y_t\right|\le\frac{1}{m}.$$

Since $\widetilde{Y}^m$ has a limit on the left at T, so does Y.
Now let us precise the behaviour of Y in a neighbourhood of T.

Lemma 8. *For $0\le t<T$, $\mathbb{P}$-a.s.*

$$Y_t\ge\mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{1}{q(T-t)+\frac{1}{\xi^q}}\right)^{1/q}\right].$$

The right hand side is obtained through the following operation: first, we solve the ordinary differential equation $y'=y^{1+q}$ with ξ as terminal condition, then we project this solution on the σ-algebra $\mathcal{F}_t$.
Proof. See proof of Lemma 11 in [83]. □

Proposition 11. *On the set $\{\xi=\infty\}$, a.s.*

$$\lim_{t\nearrow T}(T-t)^{1/q}Y_t=\left(\frac{1}{q}\right)^{1/q}$$

Proof. See proof of Proposition 13 in [83]. □
Note that if $\xi=+\infty$ a.s. we have proved that there is a unique solution (Y,Z,U) to the BSDE (3.1.1) given by $Y_t=(q(T-t))^{-1/q}$, $Z_t=U_t=0$.

3.3.2 Continuity at time T for $q>2$

In order to prove that the limit at time T of Y_t is equal to ξ, i.e.

$$\lim_{t\to T}Y_t=\xi,$$

we follow the same procedure as in [83] and we work in the Markovian setting. We suppose now our terminal condition is of the form: $\xi=g(X_T)$. The function g is defined on $\mathbb{R}^d$ with values in $\mathbb{R}_+\cup\{+\infty\}$ and we denote

$$\mathcal{S}:=\{x\in\mathbb{R}^d\quad s.t.\quad g(x)=\infty\}$$

the set of singularity points for the terminal condition induced by g. This set $\mathcal{S}$ is supposed to be closed. We also denoted by Γ the boundary of $\mathcal{S}$. The process X is the solution of a SDE with jumps:

$$X_t=X_0+\int_0^t b(s,X_s)ds+\int_0^t\sigma(s,X_s)dW_s+\int_0^t\int_E h(s,X_{s_-},e)\tilde{\mu}(de,ds). \tag{3.3.9}$$

The coefficients $b:\Omega\times[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$, $\sigma:\Omega\times[0,T]\times\mathbb{R}^d\to\mathbb{R}^{d\times k}$ and $h:\Omega\times[0,T]\times\mathbb{R}^d\times E\to\mathbb{R}^d$ satisfy

Assumptions (B):

1. b, σ and h are jointly continuous w.r.t. (t,x) and Lipschitz continuous w.r.t. x uniformly in t, e or ω, i.e. there exists a constant C such that for any $(\omega,t,e) \in \Omega\times[0,T]\times E$, for any x and y in $\mathbb{R}^d$: a.s.

$$|b(t,x)-b(t,y)| + |\sigma(t,x)-\sigma(t,y)| \leq C|x-y| \tag{B1}$$

and

$$\int_E |h(t,x,e)-h(t,y,e)|^2 \lambda(de) \leq C|x-y|. \tag{B2}$$

2. b, σ and h growth at most linearly:

$$|b(t,x)| + |\sigma(t,x)| \leq C(1+|x|). \tag{B3}$$

3. h is bounded w.r.t. t and x and there exists a constant C_h such that a.s.

$$|h(t,x,e)| \leq C_h(1\wedge|e|). \tag{B4}$$

Under these Assumptions **(B)**, the forward SDE (3.3.9) has a unique strong solution X (see [70] or [84]). Now the second hypothesis on ξ is: for all compact set $\mathcal{K}\subset\mathbb{R}^d\setminus\mathcal{S}$

$$g(X_T)\mathbf{1}_{\mathcal{K}}(X_T) \in L^1(\Omega,\mathcal{F}_T,\mathbb{P}). \tag{3.3.10}$$

We consider (Y^n,Z^n,U^n) the solution of the BSDE (3.1.1) with terminal condition $\xi\wedge n$:

$$Y^n_t = (\xi\wedge n) - \int_t^T (Y^n_s)^{q+1}ds - \int_t^T Z^n_s dW_s - \int_t^T\int_E U^n_s(e)\tilde{\mu}(de,ds).$$

Let ϕ be now a $C^2(\mathbb{R})$ bounded function with bounded derivatives. We compute Itô's formula to the process $Y^n\phi(X)$ between 0 and t.

$$\begin{aligned}
Y^n_t\phi(X_t) &= Y^n_0\phi(X_0) + \int_0^t Y^n_{s_-}d\phi(X_s) + \int_0^t \phi(X_{s_-})dY^n_s + \langle Y^n,\phi(X)\rangle_t \\
&= Y_0\phi(X_0) + \int_0^t \phi(X_{s_-})(Y_s)^{q+1}ds \\
&\quad + \int_0^t Y^n_{s_-}\left(\nabla\phi(X_s)b(s,X_s) + \frac{1}{2}\mathrm{Trace}(D^2\phi(X_s)(\sigma\sigma^*)(s,X_s))\right)ds \\
&\quad + \int_0^t\int_E Y^n_{s_-}\left(\phi(X_s)-\phi(X_{s_-})-\nabla\phi(X_{s_-})h(s,X_{s_-},e)\right)\mu(ds,de) \\
&\quad + \int_0^t Y^n_{s_-}\nabla\phi(X_s)\sigma(s,X_s)dW_s + \int_0^t \phi(X_{s_-})Z^n_s dW_s \\
&\quad + \int_0^t\int_E \phi(X_{s_-})U^n_s(e)\tilde{\mu}(de,ds) + \int_0^t\int_E Y^n_{s_-}\nabla\phi(X_s)h(s,X_{s_-},e)\tilde{\mu}(de,ds) \\
&\quad + \int_0^t \nabla\phi(X_s)\sigma(s,X_s)Z^n_s ds + \int_0^t\int_E (\phi(X_s)-\phi(X_{s_-}))U^n_s(e)\mu(ds,de).
\end{aligned}$$

Since (Z^n, U^n) and X are in $\mathbb{H}^2(0,T)$, and since ϕ and the derivatives of ϕ are supposed to be bounded, we can take the expectation of these terms:

$$\begin{aligned}
\mathbb{E}[Y_t^n \phi(X_t)] &= \mathbb{E}[Y_0^n \phi(X_0)] + \mathbb{E}\left[\int_0^t \phi(X_{s_-})(Y_s^n)^{q+1} ds\right] \\
&+ \mathbb{E}\left[\int_0^t Y_{s_-}^n \left(\nabla\phi(X_s) b(s,X_s) + \frac{1}{2}\text{Trace}(D^2\phi(X_s)\sigma^2(s,X_s))\right) ds\right] \\
&+ \mathbb{E}\left[\int_0^t \int_E Y_{s_-}^n \left(\phi(X_s) - \phi(X_{s_-}) - \nabla\phi(X_{s_-}) h(s, X_{s_-}, e)\right)\lambda(de) ds\right] \\
&+ \mathbb{E}\left[\int_0^t \nabla\phi(X_s)\sigma(s,X_s) Z_s^n ds\right] + \mathbb{E}\left[\int_0^t \int_E (\phi(X_s) - \phi(X_{s_-})) U_s^n(e)\lambda(de) ds\right].
\end{aligned} \tag{3.3.11}$$

Recall the main idea of [83]. First we prove that we can pass to the limit on n in (3.3.11) and that the limits have suitable integrability conditions on $[0,T]\times\Omega$. Secondly we write (3.3.11) between t and T and we pass to the limit when t goes to T.

Now we choose ϕ such that the support of ϕ is included in $\mathcal{R} = \mathcal{S}^c$. From the assumptions on g, we have for any n:

$$\mathbb{E}(Y_T^n \phi(X_T)) \leq \mathbb{E}(g(X_T)\phi(X_T)) < +\infty.$$

Moreover

$$\mathbb{E}(Y_0^n \phi(X_0)) \leq \frac{1}{(qT)^{1/q}} \mathbb{E}(\phi(X_0)) < +\infty.$$

We use Proposition 8 and Hölder inequality to obtain:

$$\begin{aligned}
\mathbb{E}\left[\int_0^T |\nabla\phi(X_s)\sigma(s,X_s) Z_s^n| ds\right] &\leq \left[\mathbb{E}\int_0^T (T-s)^{2/q} |Z_s^n|^2 ds\right]^{1/2} \\
&\qquad \left[\mathbb{E}\int_0^T \frac{|\nabla\phi(X_s)\sigma(s,X_s)|^2}{(T-s)^{2/q}} ds\right]^{1/2} \\
&\leq 16\left(\frac{1}{q}\right)^{2/q} \left[\mathbb{E}\int_0^T \frac{|\nabla\phi(X_s)\sigma(s,X_s)|^2}{(T-s)^{2/q}} ds\right]^{1/2} < +\infty
\end{aligned} \tag{3.3.12}$$

since $q > 2$, since $\nabla\phi$ is supposed to be bounded, σ grows linearly and $X \in \mathbb{H}^2(0,T)$. The same estimate holds for U^n:

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \int_E |(\phi(X_s) - \phi(X_{s_-})) U_s^n(e)| \lambda(de) ds\right] \\
&\leq 16\left(\frac{1}{q}\right)^{2/q} \left[\mathbb{E}\int_0^T \int_E \frac{|(\phi(X_{s^-} + h(s,X_{s^-},e)) - \phi(X_{s_-}))|^2}{(T-s)^{2/q}} \lambda(de) ds\right]^{1/2} < +\infty
\end{aligned} \tag{3.3.13}$$

Now we deal with the term:

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T Y_{s_-}^n \mathcal{L}(\phi)(s,X_s) ds\right] \\
&\quad = \mathbb{E}\left[\int_0^T Y_{s_-}^n \left(\nabla\phi(X_s) b(s,X_s) + \frac{1}{2}\text{Trace}(D^2\phi(X_s)\sigma^2(s,X_s))\right) ds\right].
\end{aligned}$$

As in [83] we consider test functions ϕ of the form ψ^α. Thus with Hölder inequality we obtain:

$$\mathbb{E}\left[\int_0^T |Y^n_{s_-}\mathcal{L}(\psi^\alpha)(s,X_s)|ds\right] \leq \left[\mathbb{E}\int_0^T \psi^\alpha(X_s)(Y^n_s)^{q+1}ds\right]^{1/(q+1)} \times \left[\mathbb{E}\int_0^T \psi^{-\alpha/q}(X_s)|\mathcal{L}(\psi^\alpha)(s,X_s)|^{(q+1)/q}ds\right]^{q/(q+1)},$$

and there exists a constant C depending only on ψ, α, σ and b such that

$$|\mathcal{L}(\psi^\alpha)| \leq C\psi^{\alpha-2}.$$

Thus for $\alpha > 2(q+1)/q$

$$\mathbb{E}\int_0^T \psi^{-\alpha/q}(X_s)|\mathcal{L}(\psi^\alpha)(s,X_s)|^{(q+1)/q}ds \leq C\mathbb{E}\int_0^T \psi^{-\alpha/q+(\alpha-2)(q+1)/q}(X_s)ds < +\infty.$$

Then

$$\mathbb{E}\left[\int_0^T |Y^n_{s_-}\mathcal{L}(\psi^\alpha)(s,X_s)|ds\right] \leq C\left[\mathbb{E}\int_0^T \psi^\alpha(X_s)(Y^n_s)^{q+1}ds\right]^{1/(q+1)}. \tag{3.3.14}$$

Therefore the main difference with [83] comes from the term

$$\mathbb{E}\left[\int_0^t\int_E Y^n_{s_-}\left(\phi(X_s)-\phi(X_{s_-})-\nabla\phi(X_{s_-})h(s,X_{s_-},e)\right)\lambda(de)ds\right]. \tag{3.3.15}$$

In order to control this term, we have to add some extra assumptions on the jumps of X and $\mathcal{S}$.

Assumption (C):

- The boundary $\partial\mathcal{S} = \Gamma$ is compact and of class C^2.
- For any $x \in \mathcal{S}$, any $s \in [0,T]$ and λ-a.s.

  $$x + h(s,x,e) \in \mathcal{S}.$$

  Furthermore there exists a constant $\kappa > 0$ such that if $x \in \Gamma$, then for any $s \in [0,T]$, $d(x+h(s,x,e),\Gamma) \geq \kappa$, λ-a.s.

These assumptions mean in particular that if $X_{s^-} \in \mathcal{S}$, then $X_s \in \mathcal{S}$ a.s. Moreover if X_{s^-} belongs to the boundary of $\mathcal{S}$, and if there is a jump at time s, then X_s is in the interior of $\mathcal{S}$.
Since Γ is compact and of class C^2, then there exists a constant $\mu_0 > 0$ such that if $\Gamma(\mu_0) := \{x \in \mathbb{R}^d : d(x,\Gamma) < \mu_0\}$, then for every $y \in \Gamma(\mu_0)$, there exists a unique $z \in \Gamma$ such that $d(y,\Gamma) = \|y-z\|$. Moreover we can define the exterior unit normal $\overrightarrow{n}(y)$ for any $y \in \Gamma$. In the following we will consider the following choice of test functions. For any $0 < \mu \leq \mu_0$, let $\theta \in C^\infty(\mathbb{R}^d)$ such that $\theta(x) \in [0,1]$, $\theta(x) > 0$ if $x \notin \Gamma(\mu/2)$, and

$$\theta(x) = \begin{cases} 0 & \text{on } \Gamma(\mu/2) \\ 1 & \text{on } \Gamma(\mu)^c. \end{cases}$$

Now let consider ϕ defined as follow:

$$\phi(x) = \begin{cases} 0 & \text{on } \mathcal{S} \\ \theta(x) & \text{on } \mathcal{S}^c = \mathcal{R}. \end{cases} \tag{3.3.16}$$

Therefore $\phi \in C_b^\infty(\mathbb{R}^d)$ (all derivatives exist and are bounded) and the support of ϕ is included in $\mathcal{R} = \mathcal{S}^c$.

Lemma 9. *Under the above assumptions, let us choose $\mu_1 < \mu_0$ such that $(1+K_h)\mu_1 < \kappa$ (K_h is the Lipschitz constant of h w.r.t. x). We have for any $0 < \mu < \mu_1$:*

$$\phi(X_{s^-}) = 0 \Rightarrow \phi(X_s) = 0.$$

Moreover

$$\frac{\phi(X_s)}{\phi(X_{s^-})} = \phi(X_s)\mathbf{1}_{\Gamma(\mu)^c}(X_{s^-}).$$

Proof. We consider the case where $X_{s_-} \notin supp(\phi)$, that is $\phi(X_{s_-}) = 0$. Thus X_{s^-} is in $\mathcal{S} \cup \Gamma(\mu/2)$.

1. If $X_{s_-} \in \mathcal{S}$, then $X_s \in \mathcal{S}$, hence $\phi(X_s) = 0$.

2. Let $z \in \mathcal{S}^c \cap \Gamma(\mu/2)$ and $x \in \Gamma$ such that $d(z, \mathcal{S}) = \|z - x\|$. Let us prove that $z + h(s, z, e) \in \mathcal{S}$ by contradiction. Assume that $z + h(s, z, e) \notin \mathcal{S}$ and consider the following convex combination:
$$z_t := (1-t)(z + h(s, z, e)) + t(x + h(s, x, e)).$$
Now since h is Lipschitz continuous w.r.t. x:
$$\begin{aligned} \|z_t - (x + h(s, x, e))\| &= (1-t)\|z + h(s, z, e) - x - h(s, x, e)\| \\ &\leq (1-t)(1+K_h)\|z - x\| \leq \frac{(1-t)(1+K_h)\mu}{2} \\ &\leq \frac{(1+K_h)\mu}{2} < \kappa. \end{aligned}$$
Since $x \in \Gamma$, $x + h(s, x, e) \in \mathcal{S}$. But $z + h(s, z, e) \notin \mathcal{S}$. Thus by continuity there exists $t_0 \in (0, 1)$ such that
$$z_{t_0} := (1-t_0)(z + h(s, z, e)) + t_0(x + h(s, x, e)) \in \Gamma.$$
Thus we have obtained $x \in \Gamma$ and $z_{t_0} \in \Gamma$ such that
$$\|z_{t_0} - (x + h(s, x, e))\| < \kappa \Rightarrow d(x + h(x, s, e), \Gamma) < \kappa.$$
This leads to a contradiction. So we deduce $z + h(s, z, e) \in \mathcal{S}$.

   Hence if $X_{s_-} \in \mathcal{S}^c \cap \Gamma(\mu/2)$, $X_s \in \mathcal{S}$ and $\phi(X_s) = 0$.

Now consider the quotient

$$\frac{\phi(X_s)}{\phi(X_{s^-})} = \frac{\phi(X_s)}{\phi(X_{s^-})}\mathbf{1}_{\mathrm{supp}(\phi)}(X_{s^-}).$$

The first part of the proof shows that for any $\mu < \mu_1$, we have:

$$\frac{\phi(X_s)}{\phi(X_{s^-})} = \phi(X_s)\mathbf{1}_{\Gamma(\mu)^c}(X_{s^-}).$$

Indeed if X_{s^-} is in $\mathrm{supp}(\phi) \cap \Gamma(\mu)$, then $X_s \in \mathcal{S}$, and thus the quotient is null. □

Now we can deal with the term given by (3.3.15). Let us take $\alpha > 2(q+1)/q$ and by Hölder inequality we obtain:

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^t \int_E Y^n_{s-} \left|\phi^\alpha(X_s) - \phi^\alpha(X_{s_-}) - \nabla(\phi^\alpha)(X_{s_-})h(s, X_{s_-}, e)\right| \lambda(de)ds\right] \\
&\leq \left[\mathbb{E}\int_0^t \phi^\alpha(X_{s^-})(Y^n_s)^{q+1} ds\right]^{\frac{1}{q+1}} \\
&\quad \times \left[\mathbb{E}\int_0^t \int_E \frac{\left|\phi^\alpha(X_s) - \phi^\alpha(X_{s_-}) - \nabla(\phi^\alpha)(X_{s_-})h(s, X_{s_-}, e)\right|^{\frac{q+1}{q}}}{\phi(X_{s_-})^{\alpha/q}} \lambda(de)ds.\right]^{\frac{q}{q+1}}
\end{aligned}
$$

The last integral is controlled by:

$$
\begin{aligned}
&\frac{\left|\phi^\alpha(X_s) - \phi^\alpha(X_{s_-}) - \nabla(\phi^\alpha)(X_{s_-})h(s, X_{s_-}, e)\right|^{\frac{q+1}{q}}}{\phi(X_{s_-})^{\alpha/q}} \\
&\leq C_q \phi^\alpha(X_s)\left(\frac{\phi(X_s)}{\phi(X_{s_-})}\right)^{\alpha/q} + C_q\phi^\alpha(X_{s^-}) \\
&\quad + C_q \phi^{\alpha-(q+1)/q}(X_{s^-})|\nabla\phi(X_{s^-})h(s, X_{s_-}, e)|.
\end{aligned}
$$

But with Lemma 9 we obtain:

$$
\begin{aligned}
&\frac{\left|\phi^\alpha(X_s) - \phi^\alpha(X_{s_-}) - \nabla(\phi^\alpha)(X_{s_-})h(s, X_{s_-}, e)\right|^{\frac{q+1}{q}}}{\phi(X_{s_-})^{\alpha/q}} \\
&\leq C_q\left[\phi^{\frac{\alpha q+1}{q}}(X_s)\mathbf{1}_{\Gamma(\mu)^c}(X_{s^-}) + \phi^\alpha(X_{s^-}) + \phi^{\alpha-\frac{q+1}{q}}(X_{s^-})|\nabla\phi(X_{s^-})h(s, X_{s_-}, e)|\right].
\end{aligned}
$$

From the assumption on ϕ and since $\alpha > 2\frac{q+1}{q}$, there exists a constant C independent on n such that:

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T \int_E Y^n_{s-} \left|\phi^\alpha(X_s) - \phi^\alpha(X_{s_-}) - \nabla(\phi^\alpha)(X_{s_-})h(s, X_{s_-}, e)\right| \lambda(de)ds\right] \\
&\leq C\left[\mathbb{E}\int_0^T \phi^\alpha(X_{s^-})(Y^n_s)^{q+1} ds\right]^{\frac{1}{q+1}}.
\end{aligned} \tag{3.3.17}
$$

Let us summarize what we obtained. For any μ small enough, any function ϕ defined by (3.3.16), and any $\alpha > 2(q+1)/q$, from (3.3.11) and using (3.3.12), (3.3.13), (3.3.14), (3.3.17) we deduce that there exists a constant C independent of n such that

$$
\mathbb{E}\int_0^T \phi^\alpha(X_{s^-})(Y^n_s)^{q+1} ds \leq C < +\infty. \tag{3.3.18}
$$

Moreover all these estimates show that we can pass to the limit in (3.3.11) (using monotone convergence theorem or dominated convergence theorem) and we have:

$$\mathbb{E}[\xi\phi^\alpha(X_T)] = \mathbb{E}[Y_t\phi^\alpha(X_t)] + \mathbb{E}\left[\int_t^T \phi^\alpha(X_{s_-})(Y_s)^{q+1}ds\right] \tag{3.3.19}$$
$$+\mathbb{E}\left[\int_t^T Y_{s_-}\mathcal{L}(\phi^\alpha)(s,X_s)ds\right]$$
$$+\mathbb{E}\left[\int_t^T\int_E Y_{s_-}\left(\phi^\alpha(X_s)-\phi^\alpha(X_{s_-})-\nabla(\phi^\alpha)(X_{s_-})h(s,X_{s_-},e)\right)\lambda(de)ds\right]$$
$$+\mathbb{E}\left[\int_t^T \nabla(\phi^\alpha)(X_s)\sigma(s,X_s)Z_sds\right]+\mathbb{E}\left[\int_t^T\int_E(\phi^\alpha(X_s)-\phi^\alpha(X_{s_-}))U_s(e)\lambda(de)ds\right].$$

Estimate (3.3.18) also holds with Y, and once again from (3.3.12), (3.3.13), (3.3.14) and(3.3.17), we can let t go to T in (3.3.19) in order to have:

$$\mathbb{E}\left[(\liminf_{t\to T} Y_t)\phi^\alpha(X_T)\right] \leq \lim_{t\to T}\mathbb{E}[Y_t\phi^\alpha(X_t)] = \mathbb{E}[\xi\phi^\alpha(X_T)].$$

Recall that the function ϕ^α is equal to one on $\mathcal{R}\cap\Gamma(\mu)^c$, that $\liminf_{t\to T} Y_t \geq \xi$ a.s. and that the limit of Y_t exists. This last inequality shows that in fact a.s.

$$\lim_{t\to T} Y_t = \xi.$$

This achieves the proof of the continuity of Y at time T.

3.4 Link with PIDE

Let us now consider the partial integro-differential equation (3.1.4). The functions b, σ and h satisfy Assumptions **(B)** and **(C)**. Moreover Π_g will denoted the space of functions $\phi : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ of polynomial growth, i.e. for some non negative constants p and C

$$\forall (t,x)\in[0,T]\times\mathbb{R}^d,\quad |\phi(t,x)|\leq C(1+|x|^p).$$

To simplify the notation, we will denote F the following function on $[0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}_d\times\mathbb{R}$:

$$F(t,x,u,p,X,c) = pb(t,x)+\frac{1}{2}\mathrm{Trace}(X(\sigma\sigma^*)(t,x))+c-u|u|^q.$$

$\mathbb{S}_d$ is the set of symmetric matrices of size $d\times d$.
For a locally bounded function u in $[0,T]\times\mathbb{R}^d$, we define its upper (resp. lower) semicontinuous envelope u^* (resp. u_*) by:

$$u^*(t,x) = \limsup_{(s,y)\to(t,x)} u(s,y) \quad (\text{resp. } u^*(t,x) = \liminf_{(s,y)\to(t,x)} u(s,y)).$$

For such equation (3.1.4) we introduce the notion of viscosity solution as in [1] (see also Definition 3.1 in [9] or Definitions 1 and 2 in [10]). Since we do not assume the continuity of the involved function u, we adapt the definition of discontinuous viscosity solution (see Definition 4.1 and 5.1 in [48]).

Definition 9. *A locally bounded function u is*

1. ***a viscosity subsolution*** *of* (3.1.4) *if it is upper semicontinuous on* $[0,T) \times \mathbb{R}^d$ *and if*

$$u(T,x) \leq g(x), \qquad x \in \mathbb{R}^d,$$

and if for any $\phi \in C^2([0,T] \times \mathbb{R}^d)$ *wherever* $(t,x) \in [0,T[\times \mathbb{R}^d$ *is a global maximum point of* $u - \phi$,

$$-\frac{\partial}{\partial t}\phi(t,x) - \mathcal{L}\phi(t,x) - \mathcal{I}\phi(t,x) + u(t,x)|u(t,x)|^q \leq 0.$$

2. ***a viscosity supersolution*** *of* (3.1.4) *if it is lower semicontinuous on* $[0,T) \times \mathbb{R}^d$ *and if*

$$u(T,x) \geq g(x), \qquad x \in \mathbb{R}^d,$$

and if for any $\phi \in C^2([0,T] \times \mathbb{R}^d)$ *wherever* $(t,x) \in [0,T[\times \mathbb{R}^d$ *is a global minimum point of* $u - \phi$,

$$-\frac{\partial}{\partial t}\phi(t,x) - \mathcal{L}\phi(t,x) - \mathcal{I}\phi(t,x) + u(t,x)|u(t,x)|^q \geq 0.$$

3. ***a viscosity solution*** *of* (3.1.4) *if its upper envelope* u^* *is a subsolution and if its lower envelope* u_* *is a supersolution of* (3.1.4).

This definition is equivalent to Definition 4.1 in [48]. Note that if a comparison principle holds for (3.1.4), then $u_* = u^*$ and thus a viscosity solution is a continuous function. We can also give another definition like Definition 5.1 in [48]. For any $\delta > 0$, the operator $\mathcal{I}$ will be split in two parts:

$$\begin{aligned}
\mathcal{I}^{1,\delta}(t,x,\phi) &= \int_{|e|\leq\delta} [\phi(x + h(t,x,e)) - \phi(x) - (\nabla\phi)(x)h(t,x,e)]\lambda(de) \\
\mathcal{I}^{2,\delta}(t,x,p,\phi) &= \int_{|e|>\delta} [\phi(x + h(t,x,e)) - \phi(x) - ph(t,x,e)]\lambda(de).
\end{aligned}$$

Definition 10. *A locally bounded and upper (resp. lower) semicontinuous function* u *is* ***a viscosity sub (resp. super) solution*** *of* (3.1.4) *if*

$$u(T,x) \leq g(x) \quad (resp.\ u(T,x) \leq g(x)), \quad x \in \mathbb{R}^d,$$

and if for any $\delta > 0$, *for any* $\phi \in C^2([0,T] \times \mathbb{R}^d)$ *wherever* $(t,x) \in [0,T[\times \mathbb{R}^d$ *is a global maximum (resp. minimum) point of* $u - \phi$ *on* $[0,T] \times B(x, R_\delta)$,

$$-\frac{\partial}{\partial t}\phi(t,x) - \mathcal{L}\phi(t,x) - \mathcal{I}^{1,\delta}(t,x,\phi) - \mathcal{I}^{2,\delta}(t,x,\nabla\phi,u) + u(t,x)|u(t,x)|^q \leq 0 \ (resp.\ \geq 0).$$

We refer to Remark 3.2 and Lemma 3.3 in [9], to condition (NLT), Proposition 1 and Section 2.2 in [10] and to Appendix in [48] for the discussion (and the proof) on the equivalence between definitions 9 and 10. The value of R_δ will depend on the function h. If h is bounded by C_h, as in [48], $R_\delta = C_h\delta$.

3.4.1 Existence of a viscosity solution with singular data

Now we introduce a function $g : \mathbb{R}^d \to \overline{R_+} = \mathbb{R}_+ \cup \{+\infty\}$ such that for any $n \in \mathbb{N}$, $x \mapsto g_n(x) = g(x) \wedge n$ is on $\mathbb{R}^d$. We define for $t \leq s \leq T$ and $x \in \mathbb{R}^d$

$$X^{t,x} = x + \int_t^s b(u, X^{t,x}_{u^-})du + \int_t^s \sigma(u, X^{t,x}_u)dW_u + \int_t^s \int_E h(u, X^{t,x}_{u^-}, e)\widetilde{\mu}(ds, de),$$

and $(Y^{n,t,x}, Z^{n,t,x}, U^{n,t,x})$ the solution of BSDE (3.1.1) with terminal condition $\xi \wedge n = g(X^{t,x}_T) \wedge n$: for $t \leq s \leq T$

$$Y^{n,t,x}_s = (\xi \wedge n) - \int_s^T Y^{n,t,x}_u |Y^{n,t,x}_u|^q du - \int_s^T Z^{n,t,x}_u dW_u - \int_s^T \int_E U^{n,t,x}_u(e)\tilde{\mu}(du, de).$$

From Theorem 3.4 and Theorem 3.5 in [9], the function $u_n(t,x) = Y^{n,t,x}_t$, $(t,x) \in [0,T] \times \mathbb{R}^d$, is the unique bounded viscosity solution of (3.1.4) with terminal condition g_n. Indeed the generator just depends on y. Even if it is not Lipschitz continuous w.r.t. y, since $Y^{n,t,x}$ is bounded by n, we can replace in the BSDE (3.1.1) our generator by f with $f(y) = -y|y|^q$ if $|y| \leq n+1$, and $f(y) = -(n+1)^q y$ if $|y| > n+1$.
Now in the previous sections, we have proved that for any $t \leq s \leq T$

$$\lim_{n \to +\infty} Y^{n,t,x}_s = Y^{t,x}_s$$

with

$$Y^{t,x}_s \leq \left(\frac{1}{q(T-s)} \right)^{1/q}.$$

Therefore the sequence $u_n(t,x)$ converges to $u(t,x)$ with

$$0 \leq u(t,x) \leq \left(\frac{1}{q(T-t)} \right)^{1/q}. \tag{3.4.20}$$

Since u_n is a continuous function, the function u is lower semi-continuous on $[0,T] \times \mathbb{R}^d$ and satisfies for all $x_0 \in \mathbb{R}^d$:

$$\liminf_{(t,x) \to (T,x_0)} u(t,x) \geq g(x_0).$$

Definition 11 (Viscosity solution with singular data)**.** *A function u is a viscosity solution of* (3.1.4) *with terminal data g if v is a viscosity solution on $[0,T[\times\mathbb{R}^d$ and satisfies:*

$$\lim_{(t,x) \to (T,x_0)} u(t,x) = g(x_0). \tag{3.4.21}$$

Theorem 18. *The function u is a viscosity solution of* (3.1.4) *with terminal data g.*

The main tool is the half-relaxed upper- and lower-limit of the sequence of functions $\{u_n\}$, i.e.

$$\overline{u}(t,x) = \limsup_{\substack{n \to +\infty \\ (t',x') \to (t,x)}} u_n(t',x') \quad \text{and} \quad \underline{u}(t,x) = \liminf_{\substack{n \to +\infty \\ (t',x') \to (t,x)}} u_n(t',x').$$

In our case, $\underline{u} = u \leq \overline{u} = u^*$ because the sequence $\{u_n\}$ is non decreasing and u_n is continuous for all $n \in \mathbb{N}^*$. Note that u^* also satisfies estimate (3.4.20).

Lemma 10. *The function u is a viscosity solution of* (3.1.4) *on* $[0,T[\times\mathbb{R}^d$.

Proof. First $u = u_* = \underline{u}$ is lower semi-continuous on $[0,T[\times\mathbb{R}^d$. From the estimate (3.4.20), for all $\delta > 0$, $n \in \mathbb{N}^*$ and all $(t,x) \in [0,T-\delta] \times \mathbb{R}^d$,

$$u_n(t,x) \leq u(t,x) \leq \left(\frac{1}{q\delta}\right)^{1/q}.$$

Since u_n is a supersolution of the PIDE (3.1.4), passing to the limit with a stability result (see the proof of Theorem 4.1 in [48] or the results in [1], [10] or [13]), we can obtain that u is a supersolution of (3.1.4) on $[0,T[\times\mathbb{R}^d$.
For convenience of the reader, let us give the main ideas (for details see the proof of Theorem 4.1 in [48]). Let $(t,x) \in [0,T[\times\mathbb{R}^d$ and let ϕ be a function which belongs to $C^{1,2}([0,T]\times\mathbb{R}^d) \cap \Pi_g$ such that $u-\phi$ has a strict global minimum in (t,x) on $[0,T]\times\mathbb{R}^d$ and we assume w.l.o.g. that $u(t,x) = \phi(t,x)$. Now let $\delta > 0$ and (t_n,x_n) be the global minimum of $u^n - \phi$ on $[0,T]\times\overline{B(x,R_\delta)}$. R_δ is a positive number such that R_δ tends to zero when $\delta \to 0$. As in [48], one can prove that

$$\lim_n (t_n,x_n) = (t,x), \quad \lim_n u^n(t_n,x_n) = u(t,x).$$

The bound (3.4.20) is crucial here. Now since u^n is a viscosity supersolution, by Definition 10,

$$\begin{aligned} -\frac{\partial}{\partial t}\phi(t_n,x_n) - \mathcal{L}\phi(t_n,x_n) - \mathcal{I}^{1,\delta}(t_n,x_n,\phi) - \mathcal{I}^{2,\delta}(t_n,x_n,\nabla\phi,u^n) \\ +u^n(t_n,x_n)|u^n(t_n,x_n)|^q \geq 0. \end{aligned} \tag{3.4.22}$$

By continuity of ϕ and h, we can pass to the limit as n goes to ∞:

$$\lim_{n\to+\infty}\left[-\frac{\partial}{\partial t}\phi(t_n,x_n) - \mathcal{L}\phi(t_n,x_n)\right] = -\frac{\partial}{\partial t}\phi(t,x) - \mathcal{L}\phi(t,x)$$

and

$$\lim_{n\to+\infty}\mathcal{I}^{1,\delta}(t_n,x_n,\phi) = \mathcal{I}^{1,\delta}(t,x,\phi).$$

Moreover

$$\lim_{n\to+\infty} u^n(t_n,x_n)|u^n(t_n,x_n)|^q = u(t,x)|u(t,x)|^q.$$

Finally since $u-\phi \geq 0$

$$\begin{aligned} &\mathcal{I}^{2,\delta}(t_n,x_n,\nabla\phi,u^n) \\ &\quad = \int_{|e|>\delta}[u^n(t_n,x_n+h(t_n,x_n,e)) - u^n(t_n,x_n) - \nabla\phi(t_n,x_n)h(t_n,x_n,e)]\lambda(de) \\ &\quad \geq \int_{|e|>\delta}[\phi(t_n,x_n+h(t_n,x_n,e)) - u^n(t_n,x_n) - \nabla\phi(t_n,x_n)h(t_n,x_n,e)]\lambda(de) \end{aligned}$$

and by Fatou's lemma

$$\liminf_{n\to+\infty}\mathcal{I}^{2,\delta}(t_n,x_n,\nabla\phi,u^n) \geq \mathcal{I}^{2,\delta}(t,x,\nabla\phi,\phi).$$

Passing to the limit in (3.4.22) we obtain:

$$-\frac{\partial}{\partial t}\phi(t,x) - \mathcal{L}\phi(t,x) - \mathcal{I}^{1,\delta}(t,x,\phi) \geq \mathcal{I}^{2,\delta}(t,x,\nabla\phi,\phi) - u(t,x)|u(t,x)|^q.$$

Thus u is a supersolution of (3.1.4) on $[0,T[\times\mathbb{R}^d$.
By the same argument we can show that u^* is a subsolution on $[0,T[\times\mathbb{R}^d$. Let $(t,x)\in[0,T)\times\mathbb{R}^d$ and $\phi\in C^{1,2}([0,T]\times\mathbb{R}^d)\cap\Pi_g$ such that $u^*-\phi$ has a strict global maximum at (t,x) on $[0,T]\times\mathbb{R}^d$ with $u^*(t,x)=\phi(t,x)$. As in [48] there exists a subsequence n_k such that

- (t_{n_k},x_{n_k}) is the global maximum of $u^{n_k}-\phi$ on $[0,T]\times\overline{B(x,R_\delta)}$;
- as k goes to ∞, $(t_{n_k},x_{n_k})\longrightarrow(t,x)$ and $u^{n_k}(t_{n_k},x_{n_k})\longrightarrow u^*(t,x)$.

Now for k large, since u^{n_k} is a subsolution, we have again by Definition 10,

$$-\frac{\partial}{\partial t}\phi(t_{n_k},x_{n_k})-\mathcal{L}\phi(t_{n_k},x_{n_k})-\mathcal{I}^{1,\delta}(t_{n_k},x_{n_k},\phi)-\mathcal{I}^{2,\delta}(t_{n_k},x_{n_k},\nabla\phi,u^{n_k}) \quad (3.4.23)$$
$$+u^{n_k}(t_{n_k},x_{n_k})|u^{n_k}(t_{n_k},x_{n_k})|^q\le 0.$$

Again since $u\le\phi$, we have

$$\begin{aligned}&\mathcal{I}^{2,\delta}(t_n,x_n,\nabla\phi,u^n)\\&=\int_{|e|>\delta}[u^n(t_n,x_n+h(t_n,x_n,e))-u^n(t_n,x_n)-\nabla\phi(t_n,x_n)h(t_n,x_n,e)]\lambda(de)\\&\le\int_{|e|>\delta}[\phi(t_n,x_n+h(t_n,x_n,e))-u^n(t_n,x_n)-\nabla\phi(t_n,x_n)h(t_n,x_n,e)]\lambda(de)\end{aligned}$$

and by continuity, Lebesgue's theorem and since $u(t,x)=\phi(t,x)$

$$-\frac{\partial}{\partial t}\phi(t,x)-\mathcal{L}\phi(t,x)-\mathcal{I}^{1,\delta}(t,x,\phi)\le\mathcal{I}^{2,\delta}(t,x,\nabla\phi,\phi)-u^*(t,x)|u^*(t,x)|^q.$$

Thus u^* is a subsolution on $[0,T[\times\mathbb{R}^d$. □
As in the case of the BSDE (3.1.1), the main difficulty is to show that

$$\limsup_{(t,x)\to(T,x_0)} u(t,x)\le g(x_0)=u(T,x_0).$$

We will prove that u^* is locally bounded on a neighbourhood of T on the set $\{g<+\infty\}$. Then, we deduce u^* is a subsolution with relaxed terminal condition and we apply this to demonstrate that $u^*(T,x)\le g(x)$ if $x\in\{g<+\infty\}$, which shows the wanted inequality on u.

Lemma 11. *Assumptions **(B)** and **(C)** hold. For any $U\subset\mathcal{R}=\{g<+\infty\}$ with a compact support, u^* is a subsolution with relaxed terminal condition:*

$$\begin{cases}-\dfrac{\partial u^*}{\partial t}-\mathcal{L}u^*-\mathcal{I}u^*+u^*|u^*|^q=0, & \text{in } [0,T[\times U;\\[2ex] \min\left[-\dfrac{\partial u^*}{\partial t}-\mathcal{L}u^*-\mathcal{I}u^*+u^*|u^*|^q;\ u^*-g\right]\le 0, & \text{in } \{T\}\times U.\end{cases}$$

Proof.
We make the same calculation as in the proof of the continuity of Y at T. Let ϕ be defined by (3.3.16). We will prove that $u_n\phi$ is uniformly bounded on $[0,T]\times\mathbb{R}^d$. On $[0,T-\delta]\times\mathbb{R}^d$ the bound (3.4.20) gives immediately the result. It remains to treat the problem on a neighbourhood of T.

We write the equality (3.3.11) between t and T, for $x \in \mathbb{R}^d$;

$$u^n(t,x)\phi(x) = \mathbb{E}[Y_T^{n,t,x}\phi(X_T^{t,x})] - \mathbb{E}\left[\int_t^T \phi(X_{s_-}^{t,x})(Y_s^{n,t,x})^{q+1}ds\right]$$
$$-\mathbb{E}\int_t^T Y_{s_-}^{n,t,x}\left[\mathcal{L}\phi(s,X_s^{t,x}) + \mathcal{I}\phi(s,X_{s^-}^{t,x})\right]ds$$
$$-\mathbb{E}\left[\int_t^T \nabla\phi(X_s^{t,x})\sigma(s,X_s^{t,x})Z_s^{n,t,x}ds\right] - \mathbb{E}\left[\int_t^T\int_E(\phi(X_s^{t,x}) - \phi(X_{s_-}^{t,x}))U_s^{n,t,x}(e)\lambda(de)ds\right].$$

The last term is controlled by:

$$\mathbb{E}\left[\int_0^T\int_E |(\phi(X_s^{t,x}) - \phi(X_{s_-}^{t,x}))U_s^{n,t,x}(e)|\lambda(de)ds\right]$$
$$\leq 16\left(\frac{1}{q}\right)^{2/q}\left[\mathbb{E}\int_0^T\int_E \frac{|(\phi(X_{s^-}+h(s,X_{s^-},e)) - \phi(X_{s_-}))|^2}{(T-s)^{2/q}}\lambda(de)ds\right]^{1/2}$$
$$\leq C = C(q,\phi).$$

Moreover

$$\left|\mathbb{E}\int_t^T Z_r^{n,t,x}.\nabla\phi(X_r^{t,x})\sigma(r,X_r^{t,x})dr\right|$$
$$\leq 16\left(\frac{1}{q}\right)^{2/q}\left(\mathbb{E}\int_t^T \frac{\|\nabla\phi(X_r^{t,x})\sigma(r,X_r^{t,x})\|^2}{(T-r)^{2/q}}dr\right)^{1/2} \leq C = C(q,\phi).$$

Here, we use the fact that $q > 2$, θ is bounded, and the condition (B1).
Thus, we have:

$$\mathbb{E}\int_t^T \phi(X_r^{t,x})(Y_r^{n,t,x})^{1+q}dr + \mathbb{E}\int_t^T Y_r^{t,x,n}\left[\mathcal{L}\phi(X_r^{t,x}) + \mathcal{I}\phi(s,X_{r^-}^{t,x})\right]dr$$
$$\leq \mathbb{E}(Y_T^{n,t,x}\phi(X_T^{t,x})) - \mathbb{E}\int_t^T Z_r^{n,t,x}.\nabla\phi(X_r^{t,x})\sigma(r,X_r^{t,x})dr$$
$$-\mathbb{E}\int_t^T\int_E(\phi(X_s^{t,x}) - \phi(X_{s_-}^{t,x}))U_s^{n,t,x}(e)\lambda(de)ds$$

The right hand side is bounded by the supremum of $g\phi$ and $2C$. In the left hand side, the second term is controlled by the first one raised to a power strictly smaller than 1 using Hölder's inequality (see (3.3.14) and (3.3.17)). Therefore, there exists a constant C independent of n, t and x:

$$\mathbb{E}\int_t^T \phi(X_r^{t,x})(Y_r^{t,x,n})^{1+q}dr \leq C.$$

We deduce that:

$$u_n(t,x)\phi(x) \leq C = C(T,g,\phi,q).$$

Hence for any $0 < \mu < \mu_1$, if $U = \mathcal{R}\cap\Gamma(\mu)^c$, u_n is uniformly bounded on $[0,T]\times U$ w.r.t. to n. Therefore, u^* is bounded on $[0,T]\times U$. We know that u_n is a subsolution of the PIDE (3.1.4) restricted to $[0,T]\times U$, i.e.

$$\begin{cases} -\dfrac{\partial u_n}{\partial t}(t,x) - \mathcal{L}u_n(t,x) + u_n(t,x)|u_n(t,x)|^q = 0, \ (t,x)\in[0,T[\times U; \\ \\ u_n(T,x) = (g\wedge n)(x), \ x\in U. \end{cases}$$

From Lemma 10, u^* is a subsolution of the PIDE (3.1.4) on $[0,T[\times U$.
The behaviour at time T is a adaptation of Theorem 4.1 in [8] (see also section 4.4.5 in [8]). Since g is continuous,

$$g(x) = \overline{g}(x) = \limsup_{\substack{n\to+\infty \\ x'\to x}} (g \wedge n)(x').$$

Now assume that for $\phi \in C^{1,2}([0,T] \times \mathbb{R}^d) \cap \Pi_g$ such that $u^* - \phi$ has a strict global maximum on $[0,T] \times U$ at (T,x) and suppose that $u^*(T,x) > g(x)$. There exists a subsequence n_k such that (t_{n_k}, x_{n_k}) is the global maximum of $u^{n_k} - \phi$ on $[0,T] \times \overline{B(x, R_\delta)}$ and as k goes to ∞, $(t_{n_k}, x_{n_k}) \longrightarrow (T,x)$ and $u^{n_k}(t_{n_k}, x_{n_k}) \longrightarrow u^*(T,x)$. This implies in particular that $t_{n_k} < T$ for any k large enough. If not, then up to a subsequence (still denoted n_k),

$$u^*(t,x) = \limsup_k u^{n_k}(t_{n_k}, x_{n_k}) = \limsup_k u^{n_k}(T, x_{n_k}) = \limsup_k (g \wedge n_k)(x_{n_k}) \leq g(x).$$

Since u^{n_k} is a subsolution, we still have (3.4.23) and passing though the limit we obtain

$$-\frac{\partial}{\partial t}\phi(T,x) - \mathcal{L}\phi(T,x) - \mathcal{I}^{1,\delta}(T,x,\phi) \leq \mathcal{I}^{2,\delta}(T,x,\nabla\phi,\phi) - u^*(T,x)|u^*(T,x)|^q.$$

Thus u^* is a subsolution on $[0,T] \times U$. □

Now Theorem 4.7 in [8] (with straightforward modifications) shows that $u^* \leq g$ in $\{T\} \times U$. In other words for any $x_0 \in \mathcal{R}$,

$$\limsup_{(t,x)\to(T,x_0)} u(t,x) \leq g(x_0).$$

With Inequality (3.4.21), we obtain the desired behaviour of u near terminal time T. This achieves the proof of Theorem 18. The next proposition precises the behaviour of the solution u on a neighbourhood of T.

Proposition 12. *The previously defined solution u satisfies for all x in the interior of $\{g = +\infty\}$:*

$$\lim_{t\to T} [q(T-t)]^{1/q} u(t,x) = 1.$$

Proof. See the proof of Proposition 22 in [83]. □

3.4.2 Minimal solution

The aim here is to prove minimality of the viscosity solution obtained by approximation among all non negative viscosity solutions (Theorem 18). We compare a viscosity solution v (in the sense of Definition 11) with u_n, for all integer n: for all $(t,x) \in [0,T] \times \mathbb{R}^d$, $u_n(t,x) \leq v_*(t,x)$. We deduce that $u \leq v_* \leq v$. Recall that $g : \mathbb{R}^d \to \overline{\mathbb{R}^+}$ is continuous, which implies that $g \wedge n : \mathbb{R}^d \to \mathbb{R}^+$ is continuous.

Proposition 13. *$u_n \leq v_*$, where v is a non negative viscosity solution of the PDE (3.1.4).*

Proof. This result seems to be a direct consequence of a well-known maximum principle for viscosity solutions (see [8] or [25] when $\mathcal{I} = 0$, [9], [10] or [48] in general). But to the best of our knowledge, this principle was not proved for solutions which can take the value $+\infty$. Thus, following the proof of Proposition 4.1 in [48] or Theorem 3 in [10], we just give here the main points.

The beginning of the proof is exactly the same as the proof of Proposition 23 in [83]. We fix $\varepsilon > 0$ and $n \geq 1$ and we define $u_{n,\varepsilon}(t,x) = u_n(t,x) - \frac{\varepsilon}{t}$. We will prove that $u_{n,\varepsilon} \leq v_*$ for every ε, hence we deduce $u_n \leq v_*$.
We suppose that there exists $(s,z) \in [0,T]\times\mathbb{R}^m$ such that $u_{n,\varepsilon}(s,z)-v_*(s,z) \geq \nu > 0$ and we will find a contradiction. First of all, it is clear that s is not equal to 0 or T, because $u_{n,\varepsilon}(0,z) = -\infty$ and $v_*(T,z) \geq g(z)$ (by definition of a supersolution).
$u_{n,\varepsilon}$ and $-v_*$ are bounded from above on $[0,T]\times\mathbb{R}^m$ respectively by n and 0. Thus, for $(\alpha,\beta) \in (\mathbb{R}^*)^2$, if we define:

$$m(t,x,y) = u_{n,\varepsilon}(t,x) - v_*(t,y) - \frac{\alpha}{2}|x-y|^2 - \beta\left(|x|^2 + |y|^2\right),$$

m has a supremum $\mu_{\alpha,\beta}$ on $[0,T]\times\mathbb{R}^m\times\mathbb{R}^m$ and the penalization terms assure that the supremum is attained at a point $(\hat{t},\hat{x},\hat{y}) = (t_{\alpha,\beta},x_{\alpha,\beta},y_{\alpha,\beta})$. By classical arguments we prove that if β is sufficiently small

$$\nu/2 \leq \mu_{\alpha,\beta}, \quad |\hat{x}|^2 + |\hat{y}|^2 \leq \frac{n}{\beta} \quad \text{and} \quad |\hat{x}-\hat{y}|^2 \leq \frac{2n}{\alpha}. \tag{3.4.24}$$

Moreover for α large enough, the time $\hat{t}$ satisfies $0 < \hat{t} < T$ (see [83] for the details).
For α large enough, we can apply Jensen-Ishii's Lemma for non local operator established by Barles and Imbert (Lemma 1 and Corollary 2 in [10]) with $u_{n,\varepsilon}$ subsolution, v_* supersolution and $\phi(x,y) = \frac{\alpha}{2}|x-y|^2 + \beta\left(|x|^2+|y|^2\right)$ at the point $(\hat{t},\hat{x},\hat{y})$. For any $\delta > 0$ there exists $\gamma > 0$ and (a,p,X), (b,q,Y) such that

- $a = b$, $p = \nabla_x\phi(\hat{x},\hat{y}) = \alpha(\hat{x}-\hat{y}) + 2\beta\hat{x}$, $q = -\nabla_y\phi(\hat{x},\hat{y}) = -\alpha(\hat{y}-\hat{x}) - 2\beta\hat{y}$
- X and Y are symmetric matrices of size $d\times d$ such that
$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\beta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + o_\gamma(1)$$
- $$\begin{aligned} -a - F(\hat{t},\hat{x},u_{n,\varepsilon}(t,\hat{x}),p,X,\mathcal{I}^{1,\delta}(\hat{t},\hat{x},\phi_\gamma(.,\hat{y})) + \mathcal{I}^{2,\delta}(\hat{t},\hat{x},p,u_{n,\varepsilon}(\hat{t},\hat{x}))) &\leq -\frac{\varepsilon}{T^2} \\ -b - F(\hat{t},\hat{y},v_*(t,\hat{y}),q,Y,\mathcal{I}^{1,\delta}(\hat{t},\hat{y},-\phi_\gamma(\hat{x},.)) + \mathcal{I}^{2,\delta}(\hat{t},\hat{y},q,v_*(\hat{t},\hat{y}))) &\geq 0. \end{aligned}$$

The result holds for any $0 < \gamma < \bar{\gamma}$ and the value $\bar{\gamma} > 0$ depends on the coefficients of the PIDE. The function ϕ_γ is defined in the same way as in [10]. Proposition 3 in [10] shows that we can replace ϕ_γ in $\mathcal{I}^{1,\delta}$ by ϕ up to some $o_\gamma(1)$. We substract the two previous inequalities:

$$\begin{aligned} \frac{\varepsilon}{T^2} + o_\gamma(1) \leq\; & F(\hat{t},\hat{x},u_{n,\varepsilon}(t,\hat{x}),p,X,\mathcal{I}^{1,\delta}(\hat{t},\hat{x},\phi(.,\hat{y})) + \mathcal{I}^{2,\delta}(\hat{t},\hat{x},p,u_{n,\varepsilon}(\hat{t},\hat{x}))) \\ & -F(\hat{t},\hat{y},v_*(t,\hat{y}),q,Y,\mathcal{I}^{1,\delta}(\hat{t},\hat{y},-\phi(\hat{x},.)) + \mathcal{I}^{2,\delta}(\hat{t},\hat{y},q,v_*(\hat{t},\hat{y}))). \end{aligned} \tag{3.4.25}$$

Let us separate the local terms with the non local ones. For the first ones we have:

$$\begin{aligned} & \frac{1}{2}\mathrm{Trace}\left(\sigma\sigma^*(\hat{t},\hat{x})X\right) - \frac{1}{2}\mathrm{Trace}\left(\sigma\sigma^*(\hat{t},\hat{y})Y\right) \\ & + \left(b(\hat{t},\hat{x}) - b(\hat{t},\hat{y})\right).\alpha(\hat{x}-\hat{y}) + 2\beta\left(b(\hat{t},\hat{x}).\hat{x} + b(\hat{t},\hat{y}).\hat{y}\right) \\ & -u_{n,\varepsilon}(\hat{t},\hat{x})^{1+q} + v_*(\hat{t},\hat{y})^{1+q}. \end{aligned}$$

As in [83] we prove that there exists a constant K independent of α and β such that:

$$\begin{aligned}&\left(b(\hat{t},\hat{x})-b(\hat{t},\hat{y})\right).\alpha(\hat{x}-\hat{y})+2\beta\left(b(\hat{t},\hat{x}).\hat{x}+b(\hat{t},\hat{y}).\hat{y}\right)\\&\quad\le \alpha K|\hat{x}-\hat{y}|^2+2\beta K(1+|\hat{x}|^2+|\hat{y}|^2),\end{aligned}\tag{3.4.26}$$

and

$$\text{Trace}\left(\sigma\sigma^*(\hat{t},\hat{x})X\right)-\text{Trace}\left(\sigma\sigma^*(\hat{t},\hat{y})Y\right)\le K\alpha|\hat{x}-\hat{y}|^2+K\beta(1+|\hat{x}|^2+|\hat{y}|^2).\tag{3.4.27}$$

Moreover

$$v_*(\hat{t},\hat{y})\le u_{n,\varepsilon}(\hat{t},\hat{x}).\tag{3.4.28}$$

Now we deal with the non local terms. First we control

$$\begin{aligned}&\mathcal{I}^{2,\delta}(\hat{t},\hat{x},p,u_{n,\varepsilon}(\hat{t},\hat{x}))-\mathcal{I}^{2,\delta}(\hat{t},\hat{y},q,v_*(\hat{t},\hat{y}))\\&=\int_{|e|>\delta}[u_{n,\varepsilon}(\hat{x}+h(\hat{t},\hat{x},e))-u_{n,\varepsilon}(\hat{t},\hat{x})-ph(\hat{t},\hat{x},e)]\lambda(de)\\&\quad-\int_{|e|>\delta}[v_*(\hat{y}+h(\hat{t},\hat{y},e))-v_*(\hat{t},\hat{y})-qh(\hat{t},\hat{y},e)]\lambda(de)\end{aligned}$$

We use the following inequality:

$$\begin{aligned}&u_{n,\varepsilon}(\hat{x}+h(\hat{t},\hat{x},e))-v_*(\hat{y}+h(\hat{t},\hat{y},e))\le m(\hat{t},\hat{x},\hat{y})\\&\quad+\frac{\alpha}{2}|\hat{x}+h(\hat{t},\hat{x},e)-\hat{y}-h(\hat{t},\hat{y},e)|^2+\beta\left(|\hat{x}+h(\hat{t},\hat{x},e)|^2+|\hat{y}+h(\hat{t},\hat{y},e)|^2\right)\\&\le m(\hat{t},\hat{x},\hat{y})+\frac{\alpha}{2}|\hat{x}-\hat{y}|^2+\beta\left(|\hat{x}|^2+|\hat{y}|^2\right)\\&\quad+\frac{\alpha}{2}|h(\hat{t},\hat{x},e)-h(\hat{t},\hat{y},e)|^2+\beta\left(|h(\hat{t},\hat{x},e)|^2+|h(\hat{t},\hat{y},e)|^2\right)\\&\quad+\alpha(\hat{x}-\hat{y})(h(\hat{t},\hat{x},e)-h(\hat{t},\hat{y},e))+2\beta\left(\hat{x}h(\hat{t},\hat{x},e)+\hat{y}h(\hat{t},\hat{y},e)\right)\\&=u_{n,\varepsilon}(\hat{t},\hat{x})-v_*(\hat{t},\hat{y})+ph(\hat{t},\hat{x},e)-qh(\hat{t},\hat{y},e)\\&\quad+\frac{\alpha}{2}|h(\hat{t},\hat{x},e)-h(\hat{t},\hat{y},e)|^2+\beta\left(|h(\hat{t},\hat{x},e)|^2+|h(\hat{t},\hat{y},e)|^2\right).\end{aligned}$$

Therefore by Assumption **(C)** on h, there exists K independent of α and β such that:

$$\mathcal{I}^{2,\delta}(\hat{t},\hat{x},p,u_{n,\varepsilon}(\hat{t},\hat{x}))-\mathcal{I}^{2,\delta}(\hat{t},\hat{y},q,v_*(\hat{t},\hat{y}))\le K\left(\frac{\alpha}{2}|\hat{x}-\hat{y}|^2+\beta\right).\tag{3.4.29}$$

Now

$$\begin{aligned}&\mathcal{I}^{1,\delta}(\hat{t},\hat{x},\phi(.,\hat{y}))-\mathcal{I}^{1,\delta}(\hat{t},\hat{x},-\phi(\hat{x},.))\\&=\int_{|e|\le\delta}[\phi(\hat{x}+h(\hat{t},\hat{x},e),\hat{y})-\phi(\hat{x},\hat{y})-(\nabla_x\phi)(\hat{x},\hat{y})h(\hat{t},\hat{x},e)]\lambda(de)\\&\quad-\int_{|e|\le\delta}[-\phi(\hat{x},\hat{y}+h(\hat{t},\hat{y},e))+\phi(\hat{x},\hat{y})+(\nabla_y\phi)(\hat{x},\hat{y})h(\hat{t},\hat{y},e)]\lambda(de)\\&=\left(\frac{\alpha}{2}+\beta\right)\int_{|e|\le\delta}\left(|h(\hat{t},\hat{x},e)|^2+|h(\hat{t},\hat{y},e)|^2\right)\lambda(de)\\&\le 2K^2\left(\frac{\alpha}{2}+\beta\right)\int_{|e|\le\delta}(1\wedge|e|^2)\lambda(de),\end{aligned}\tag{3.4.30}$$

by Assumption (B4).
Finally plugging (3.4.26), (3.4.27), (3.4.28), (3.4.29) and (3.4.30) in (3.4.25) we obtain:

$$\frac{\varepsilon}{T^2} + o_\gamma(1) \leq K\left(\alpha|\hat{x}-\hat{y}|^2 + \beta(1+|\hat{x}|^2+|\hat{y}|^2)\right) + \left(\frac{\alpha}{2}+\beta\right) o_\delta(1), \tag{3.4.31}$$

where C is a constant independent of α and β. We let γ and δ go to zero and since

$$\lim_{\alpha\to+\infty}\lim_{\beta\to 0}\frac{\alpha}{2}|\hat{x}-\hat{y}|^2 + \beta\left(|\hat{x}|^2+|\hat{y}|^2\right) = 0,$$

the inequality (3.4.31) leads to a contradiction taking β sufficiently small and α sufficiently large. Hence $u_{n,\varepsilon} \leq v_*$ and it is true for every $\varepsilon > 0$, so the result is proved.

□

Note that if in Definition 11 we add the condition that u is bounded on $[0, T-\delta]\times\mathbb{R}^d$ for any $\delta > 0$, then we can prove that

- $u(t) \leq (q(T-t))^{-1/q}$.
- If $g \equiv +\infty$, then there is a unique viscosity solution u.

3.4.3 Regularity of the minimal solution

The function u is the minimal non negative viscosity solution of the PDE (3.1.4). We know that u is finite on $[0, T[\times\mathbb{R}^d$ (see (3.4.20)). For $\delta > 0$, u is bounded on $[0, T-\delta]\times\mathbb{R}^d$ by a constant which depends only on δ. To have more regularity on u we add some conditions on the coefficients.

1. σ and b are bounded: there exists a constant C s.t.
$$\forall (t,x)\in[0,T]\times\mathbb{R}^d, \quad |b(t,x)| + \|\sigma(t,x)\| \leq C; \tag{A5}$$
2. $\sigma\sigma^*$ is uniformly elliptic, i.e. there exists $\lambda > 0$ s.t. for all $(t,x)\in[0,T]\times\mathbb{R}^d$:
$$\forall y\in\mathbb{R}^d, \ \sigma\sigma^*(t,x)y.y \geq \lambda|y|^2. \tag{A6}$$

Proposition 14. *If λ is finite and if the coefficients of the operator $\mathcal{L}$ satisfy Conditions **(A)** and (A5) and (A6), and are Hölder-continuous in time, then u is continuous on $[0,T]\times\mathbb{R}^d$, and for all $\delta > 0$:*

$$u \in C^{1,2}([0, T-\delta]\times\mathbb{R}^d; \mathbb{R}^+). \tag{3.4.32}$$

Proof. The proof of Proposition 13 shows that there is a unique bounded and continuous viscosity solution of the Cauchy problem:

$$\begin{cases} \partial_t v + \mathcal{L}v + \mathcal{I}(v) - v|v|^q &= 0, \quad \text{on } [0, T-\delta]\times\mathbb{R}^d, \\ v(T-\delta, x) &= \phi(x) \quad \text{on } \mathbb{R}^d \end{cases} \tag{3.4.33}$$

where ϕ is supposed bounded and continuous on $\mathbb{R}^d$.
Moreover, the Cauchy problem (3.4.33) has a classical solution for every bounded and continuous function ϕ (see Lemma 12 below).

Recall that u_n is jointly continuous in (t,x) and on $[0,T-\delta]\times\mathbb{R}^d$, u_n is bounded by:

$$0 \le u_n(t,x) \le \left(\frac{1}{q\delta}\right)^{1/q}.$$

Thus, the problem (3.4.33) with condition $\phi = u_n(T-\delta,.)$ has a bounded classical solution. Since every classical solution is a viscosity solution and since u_n is the unique bounded and continuous viscosity solution of (3.4.33), we deduce that:

$$\forall \delta > 0,\ u_n \in C^{1,2}([0,T-\delta[\times\mathbb{R}^d;\mathbb{R}^+).$$

From the construction of the classical solution u_n, we also know that the sequence $\{u_n\}$ is locally bounded in $C^{\alpha,1+\alpha}([0,T-\delta/2]\times\mathbb{R}^d)$. The bound is given by the L^∞ norm of u_n which is smaller than $(T-\delta/4)^{-1/q}$. Therefore u is continuous on $[T-\delta/2]\times\mathbb{R}^d$ and if we consider the problem (3.4.33) with continuous terminal data $u(T-\delta,.)$, with the same argument as for u_n, we obtain that u is a classical solution, i.e. $u \in C^{1,2}([0,T-\delta]\times\mathbb{R}^d;\mathbb{R}^+)$.
In particular, u is continuous on $[0,T[\times\mathbb{R}^d$. And the terminal condition on u in Theorem 18 shows that u is continuous at time T. □

Lemma 12. *For every bounded and continuous function ϕ, the Cauchy problem (3.4.33) has a classical bounded solution v.*

Proof. We will use the scheme done in [83] (Proposition 24) and in Pham [80] (Proposition 5.3). For a given continuous and bounded function ϕ, the problem (3.4.33) has a unique bounded and continuous viscosity solution v (just apply Theorem 3.4 and Theorem 3.5 in [9]). We consider the following Cauchy problem: on $[0,T-\delta]\times\mathbb{R}^d$

$$\partial_t u + \widetilde{\mathcal{L}} u = f_v, \tag{3.4.34}$$

with

- $\widetilde{\mathcal{L}}$ a differential operator:

$$\widetilde{\mathcal{L}}\phi = \frac{1}{2}\mathrm{Trace}(D^2\phi\sigma\sigma^*(t,x)) + \widetilde{b}(t,x)\nabla\phi,$$

  and

$$\widetilde{b}(t,x) = b(t,x) - \int_E h(t,x,e)\lambda(de).$$

- f_v is the following function:

$$f_v(t,x) = v(t,x)|v(t,x)|^q - \int_E [v(x+h(t,x,e)) - v(x)]\lambda(de).$$

First note that v is also the unique bounded viscosity solution of (3.4.34). Moreover the assumptions (B4) and (A5) imply that since λ is finite and v is bounded, the drift term $\widetilde{b}$ and the function f_v are also bounded.
Using the result of Veretennikov [94], Theorem 3.1, the problem (3.4.34) has a unique solution u in the class $C([0,T-\delta]\times\mathbb{R}^d;\mathbb{R}^+)\cap\bigcap_{p>1} W^{1,2}_{p,loc}([0,T-\delta[\times\mathbb{R}^d)$. Now we define three processes for all $s \ge t$:

$$\overline{X}^{t,x}_s = x + \int_t^s \widetilde{b}(u,\overline{X}^{t,x}_u)du + \int_t^s \sigma(u,\overline{X}^{t,x}_u)dB_u,$$

and
$$\overline{Y}^{t,x}_s = u(s, \overline{X}^{t,x}_s), \text{ and } \overline{Z}^{t,x}_s = \nabla u(s, X^{t,x}_s)\sigma(s, \overline{X}^{t,x}_s).$$
We can apply the Itô formula to the function u (see [54], section 2.10). We have for all $s \geq t$:
$$\overline{Y}^{t,x}_s = \overline{Y}^{t,x}_{T-\delta} - \int_s^{T-\delta} f_v(r, \overline{X}^{t,x}_r)dr - \int_s^{T-\delta} \overline{Z}^{t,x}_r dB_r.$$
Since f_v is bounded, it is well known (see [72]) that the function $(t,x) \mapsto \overline{Y}^{t,x}_t$ is a continuous and bounded viscosity solution of (3.4.34). Therefore $u = v$ and thus the viscosity solution v belongs to $\bigcap_{p>1} W^{1,2}_{p,loc}([0, T-\delta[\times \mathbb{R}^d)$.
Hence for all $\alpha < 1$, v belongs to the space $H^{\beta,1+\beta}$ (the set of functions which are β-Hölder-continuous in time and $1+\beta$-Hölder-continuous in space), and the Hölder norm of v depends just on the L^∞ bound of v. Thus f_v is also Hölder continuous, and from the existence result of [58] (see section IV, theorems 5.1 and 10.1), v is a classical solution of (3.4.34) and of (3.4.33).

□

Chapter 4

Second-order BSDEs with general reflection and game options under uncertainty

Contents

4.1 Introduction

The first aim of this chapter is to extend the results of [66] to the case of doubly reflected second-order BSDEs when we assume enough regularity on one of the barriers (as in [26]) and that the two barriers are completely separated (as in [42] and [43]). In that case, we show that the right way to define a solution is to consider a 2BSDE where we add a process V which has only bounded variations (see Definition 4.2.8). Our next step towards a theory of existence and uniqueness is then to understand as much as possible how and when this bounded variation process acts. Our key result is obtained in Proposition 4.3.5, and allows us to obtain a special Jordan decomposition for V, in the sense that we can decompose it into the difference of two non-decreasing processes which never act at the same time. Thanks to this result, we are then able to obtain a priori estimates and a uniqueness result. Next, we reuse the methodology of [66] to construct a solution.

We also show that these objects are related to non-standard optimal stopping games, thus generalizing the connection between DRBSDEs and Dynkin games first proved by Cvitanić and Karatzas [27]. Finally, we show that the second order DRBSDEs allow to obtain super and sub-hedging prices for American game options (also called Israeli options) in financial markets with volatility uncertainty and that, under a technical assumption, they provide solutions of what we call uncertain Dynkin games.

The chapter is organized as follows. After recalling some notations and definitions, in Section 4.2, we treat the problem of uniqueness in Section 4.3. Section 4.4 is then devoted to the pathwise construction of a solution, thus solving the existence problem. Finally, we investigate in Section 4.5 the aforementioned game theoretical and financial applications. The Appendix is devoted to some technical results used throughout the chapter.

4.2 Definitions and Notations

4.2.1 The stochastic framework

Let $\Omega := \left\{\omega \in C([0,T],\mathbb{R}^d) : \omega_0 = 0\right\}$ be the canonical space equipped with the uniform norm $\|\omega\|_\infty := \sup_{0\leq t\leq T} |\omega_t|$, B the canonical process, $\mathbb{P}_0$ the Wiener measure, $\mathbb{F} := \{\mathcal{F}_t\}_{0\leq t\leq T}$ the filtration generated by B, and $\mathbb{F}^+ := \left\{\mathcal{F}_t^+\right\}_{0\leq t\leq T}$ the right limit of $\mathbb{F}$. A probability measure $\mathbb{P}$ will be called a local martingale measure if the canonical process B is a local martingale under $\mathbb{P}$. Then, using results of Bichteler [14] (see also Karandikar [50] for a modern account), the quadratic variation $\langle B\rangle$ and its density $\widehat{a}$ can be defined pathwise, and such that they coincide with the usual definitions under any local martingale measure.

With the intuition of modeling, volatility uncertainty, we let $\overline{\mathcal{P}}_W$ denote the set of all local martingale measures $\mathbb{P}$ such that

$$\langle B\rangle \text{ is absolutely continuous in } t \text{ and } \widehat{a} \text{ takes values in } \mathbb{S}_d^{>0},\ \mathbb{P}-a.s., \tag{4.2.1}$$

where $\mathbb{S}_d^{>0}$ denotes the space of all $d\times d$ real valued positive definite matrices.

However, since this set is too large for our purpose (in particular there are examples of measures in $\overline{\mathcal{P}}_W$ which do not satisfy the martingale representation property, see [88] for more details), we will concentrate on the following subclass $\mathcal{P}_S$ consisting of

$$\mathbb{P}^\alpha := \mathbb{P}_0\circ (X^\alpha)^{-1} \text{ where } X_t^\alpha := \int_0^t \alpha_s^{1/2}dB_s,\ t\in[0,T],\ \mathbb{P}_0-a.s., \tag{4.2.2}$$

for some $\mathbb{F}$-progressively measurable process α taking values in $\mathbb{S}_d^{>0}$ with $\int_0^T |\alpha_t|dt < +\infty$, $\mathbb{P}_0$$-a.s.$

This subset has the convenient property that all its elements do satisfy the martingale representation property and the Blumenthal $0-1$ law (see [88] for details) which are crucial tools for the BSDE theory.

4.2.2 Generator and measures

We consider a map $H_t(\omega, y, z, \gamma) : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times D_H \to \mathbb{R}$, where $D_H \subset \mathbb{R}^{d\times d}$ is a given subset containing 0, whose Fenchel transform w.r.t. γ is denoted by

$$F_t(\omega, y, z, a) := \sup_{\gamma\in D_H} \left\{\frac{1}{2}\mathrm{Tr}(a\gamma) - H_t(\omega, y, z, \gamma)\right\} \text{ for } a \in \mathbb{S}_d^{>0},$$
$$\widehat{F}_t(y, z) := F_t(y, z, \widehat{a}_t) \text{ and } \widehat{F}_t^0 := \widehat{F}_t(0, 0).$$

We denote by $D_{F_t(y,z)} := \{a,\ F_t(\omega, y, z, a) < +\infty\}$ the domain of F in a for a fixed (t, ω, y, z). As in [89] we fix a constant $\kappa \in (1, 2]$ and restrict the probability measures in $\mathcal{P}_H^\kappa \subset \overline{\mathcal{P}}_S$

Definition 4.2.1. *$\mathcal{P}_H^\kappa$ consists of all $\mathbb{P} \in \overline{\mathcal{P}}_S$ such that*

$$\underline{a}^{\mathbb{P}} \leq \widehat{a} \leq \bar{a}^{\mathbb{P}},\ dt \times d\mathbb{P} - a.s. \text{ for some } \underline{a}^{\mathbb{P}}, \bar{a}^{\mathbb{P}} \in \mathbb{S}_d^{>0},\ \text{and } \phi_H^{2,\kappa} < +\infty,$$

where

$$\phi_H^{2,\kappa} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}_{0\leq t\leq T}^{\mathbb{P}}\left(\mathbb{E}_t^{H,\mathbb{P}}\left[\int_0^T |\hat{F}_s^0|^\kappa ds\right]\right)^{\frac{2}{\kappa}}\right].$$

Definition 4.2.2. *We say that a property holds $\mathcal{P}_H^\kappa$-quasi-surely ($\mathcal{P}_H^\kappa$-q.s. for short) if it holds $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}_H^\kappa$.*

We now state the main assumptions on the function F which will be our main interest in the sequel

Assumptions (D):

(i) The domain $D_{F_t(y,z)} = D_{F_t}$ is independent of (ω, y, z).

(ii) For fixed (y, z, a), F is $\mathbb{F}$-progressively measurable in D_{F_t}.

(iii) We have the following uniform Lipschitz-type property in y and z

$$\forall (y, y', z, z', t, a, \omega),\ \left|F_t(\omega, y, z, a) - F_t(\omega, y', z', a)\right| \leq C\left(\left|y - y'\right| + \left|a^{1/2}\left(z - z'\right)\right|\right).$$

(iv) F is uniformly continuous in ω for the $||\cdot||_\infty$ norm.

(v) $\mathcal{P}_H^\kappa$ is not empty.

Remark 4.2.3. *Assumptions (ii), (iii) are completely standard in the BSDE literature since the paper [73]. Similarly, (i) was already present in the first paper on 2BSDEs in a quasi-sure formulation [89] and is linked to the fact that one does not know how to treat coupled second-order FBSDEs. The last hypothesis (iv) is also proper to the second order framework, and allows us to not only give a pathwise construction for the solution to the 2RBSDE, but to recover the very important dynamic programming property. We refer the reader to Section 4.4 for more details.*

4.2.3 Quasi-sure norms and spaces

The following spaces and the corresponding norms will be used throughout the paper. With the exception of the space $\mathbb{L}_H^{p,\kappa}$, they all are immediate extensions of the usual spaces to the quasi-sure setting.

For $p \geq 1$, $L_H^{p,\kappa}$ denotes the space of all $\mathcal{F}_T$-measurable scalar r.v. ξ with

$$\|\xi\|_{L_H^{p,\kappa}}^p := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[|\xi|^p\right] < +\infty.$$

$\mathbb{H}_H^{p,\kappa}$ denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}^d$-valued processes Z with

$$\|Z\|_{\mathbb{H}_H^{p,\kappa}}^p := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T |\widehat{a}_t^{1/2} Z_t|^2 dt\right)^{\frac{p}{2}}\right] < +\infty.$$

$\mathbb{D}_H^{p,\kappa}$ denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}$-valued processes Y with

$$\mathcal{P}_H^\kappa - q.s. \text{ càdlàg paths, and } \|Y\|_{\mathbb{D}_H^{p,\kappa}}^p := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T} |Y_t|^p\right] < +\infty,$$

where càdlàg is the French acronym for "right-continuous with left-limits".

$\mathbb{I}_H^{p,\kappa}$ denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}$-valued processes K null at 0 with

$$\mathcal{P}_H^\kappa - q.s., \text{ càdlàg and non-decreasing paths, and } \|K\|_{\mathbb{I}_H^{p,\kappa}}^p := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[(K_T)^p\right] < +\infty.$$

$\mathbb{V}_H^{p,\kappa}$ denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}$-valued processes V null at 0 with paths which are $\mathcal{P}_H^\kappa - q.s.$ càdlàg and of bounded variation, and such that

$$\|V\|_{\mathbb{V}_H^{p,\kappa}}^p := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[(\mathrm{Var_{0,T}(V)})^p\right] < +\infty.$$

For each $\xi \in L_H^{1,\kappa}$, $\mathbb{P} \in \mathcal{P}_H^\kappa$ and $t \in [0,T]$ denote

$$\mathbb{E}_t^{H,\mathbb{P}}[\xi] := \underset{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'}[\xi] \text{ where } \mathcal{P}_H^\kappa(t^+,\mathbb{P}) := \left\{\mathbb{P}' \in \mathcal{P}_H^\kappa : \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_t^+\right\}.$$

Here $\mathbb{E}_t^{\mathbb{P}}[\xi] := E^{\mathbb{P}}[\xi|\mathcal{F}_t]$. Then we define for each $p \geq \kappa$,

$$\mathbb{L}_H^{p,\kappa} := \left\{\xi \in L_H^{p,\kappa} : \|\xi\|_{\mathbb{L}_H^{p,\kappa}} < +\infty\right\} \text{ where } \|\xi\|_{\mathbb{L}_H^{p,\kappa}}^p := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\underset{0\leq t\leq T}{\operatorname{ess\,sup}^{\mathbb{P}}} \left(\mathbb{E}_t^{H,\mathbb{P}}[|\xi|^\kappa]\right)^{\frac{p}{\kappa}}\right].$$

We denote by $\mathrm{UC}_b(\Omega)$ the collection of all bounded and uniformly continuous maps $\xi : \Omega \to \mathbb{R}$ with respect to the $\|\cdot\|_\infty$-norm, and we let

$$\mathcal{L}_H^{p,\kappa} := \text{the closure of } \mathrm{UC}_b(\Omega) \text{ under the norm } \|\cdot\|_{\mathbb{L}_H^{p,\kappa}}, \text{ for every } 1 \leq \kappa \leq p.$$

Finally, for every $\mathbb{P} \in \mathcal{P}_H^\kappa$, and for any $p \geq 1$, $L^p(\mathbb{P})$, $\mathbb{H}^p(\mathbb{P})$, $\mathbb{D}^p(\mathbb{P})$, $\mathbb{I}^p(\mathbb{P})$ and $\mathbb{V}^p(\mathbb{P})$ will denote the corresponding usual spaces when there is only one measure $\mathbb{P}$.

4.2.4 Obstacles and definition

First, we consider a process S which will play the role of the upper obstacle. We will always assume that S verifies the following properties

Assumptions (E):

(i) S is $\mathbb{F}$-progressively measurable and càdlàg.

(ii) S is uniformly continuous in ω in the sense that for all t

$$|S_t(\omega) - S_t(\widetilde{\omega})| \leq \rho\left(\|\omega - \widetilde{\omega}\|_t\right), \ \forall\ (\omega, \widetilde{\omega}) \in \Omega^2,$$

for some modulus of continuity ρ and where we define $\|\omega\|_t := \sup_{0\leq s\leq t} |\omega(s)|$.

(iii) S is a semimartingale for every $\mathbb{P} \in \mathcal{P}_H^\kappa$, with the decomposition

$$S_t = S_0 + \int_0^t P_s dB_s + A_t^{\mathbb{P}},\ \mathbb{P}-a.s.,\ \text{for all } \mathbb{P} \in \mathcal{P}_H^\kappa, \tag{4.2.3}$$

where the $A^{\mathbb{P}}$ are bounded variation processes with Jordan decomposition $A^{\mathbb{P},+} - A^{\mathbb{P},-}$ and

$$\zeta_H^{2,\kappa} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \left(\mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}^{\mathbb{P}}_{0\leq t\leq T} \mathbb{E}_t^{H,\mathbb{P}}\left[\left(\int_t^T \left|\widehat{a}_s^{1/2} P_s\right|^2 ds\right)^{\kappa/2} + \left(A_T^{\mathbb{P},+}\right)^\kappa\right]\right]\right)^2 < +\infty.$$

(iv) S satisfies the following integrability condition

$$\psi_H^{2,\kappa} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}^{\mathbb{P}}_{0\leq t\leq T} \left(\mathbb{E}_t^{H,\mathbb{P}}\left[\sup_{0\leq s\leq T} |S_s|^\kappa\right]\right)^{\frac{2}{\kappa}}\right] < +\infty.$$

Remark 4.2.4. *We assumed here that S was a semimartingale. This is directly linked to the fact that this is one of the conditions under which existence and uniqueness of solutions to standard doubly reflected BSDEs with upper obstacle S are guaranteed. More precisely, this assumption is needed for us in the proof of Lemma (4.6.11), and it will be also crucial in order to obtain a priori estimates for 2BSDEs with two obstacles. This assumption is at the heart of our approach, and our proofs no longer work without it. Notice however that such an assumption was not needed for the lower obstacles considered in [66]. This is the first manifestation of an effect that we will highlight throughout the paper, namely that there is absolutely no symmetry between lower and upper obstacles in the second-order framework.*

Remark 4.2.5. *The decomposition (4.2.3) is not restrictive. Indeed, with the integrability assumption (iv), we know that for each $\mathbb{P} \in \mathcal{P}_H^\kappa$, there exists a $\mathbb{P}$-martingale $M^{\mathbb{P}}$ and a bounded variation process $A^{\mathbb{P}}$ such that*

$$S_t = S_0 + M_t^{\mathbb{P}} + A_t^{\mathbb{P}},\ \mathbb{P}-a.s.$$

Then, using the martingale representation theorem, there exists some $P_t^{\mathbb{P}} \in \mathbb{H}^2(\mathbb{P})$ such that

$$M_t^{\mathbb{P}} = \int_0^t P_s^{\mathbb{P}} dB_s.$$

Then, since S is càdlàg, by Karandikar [50], we can aggregate the family $(P^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_H^\kappa}$ into a universal process P, which gives us the decomposition (4.2.3).

Next, we also consider a lower obstacle L which will be assumed to verify

Assumptions (F):

(i) L is a $\mathbb{F}$-progressively measurable càdlàg process.

(ii) L is uniformly continuous in ω in the sense that for all t and for some modulus of continuity ρ

$$|L_t(\omega) - L_t(\widetilde{\omega})| \leq \rho\left(\|\omega - \widetilde{\omega}\|_t\right),\ \forall\ (\omega, \widetilde{\omega}) \in \Omega^2.$$

(iii) For all $t \in [0, T]$, we have

$$L_t < S_t \text{ and } L_{t^-} < S_{t^-},\ \mathcal{P}_H^\kappa - q.s.$$

(iv) We have the following integrability condition

$$\varphi_H^{2,\kappa} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}_{0\leq t\leq T}^{\mathbb{P}}\left(\mathbb{E}_t^{H,\mathbb{P}}\left[\left(\sup_{0\leq s\leq T}(L_s)^+\right)^\kappa\right]\right)^{\frac{2}{\kappa}}\right] < +\infty. \tag{4.2.4}$$

Remark 4.2.6. *Unlike for S, we did not assume here that L was a semimartingale, and we cannot interchange the roles of S and L, that is to say that the wellposedness results do not hold if we assume that L is a semimartingale instead of S.*

We shall consider the following second order doubly reflected BSDE (2DRBSDE for short) with upper obstacle S and lower obstacle L

$$Y_t = \xi + \int_t^T \widehat{F}_s(Y_s, Z_s)ds - \int_t^T Z_s dB_s + V_T - V_t,\ 0 \leq t \leq T,\ \mathcal{P}_H^\kappa - q.s. \tag{4.2.5}$$

In order to give the definition of the 2DRBSDE, we first need to introduce the corresponding standard doubly reflected BSDEs. Hence, for any $\mathbb{P} \in \mathcal{P}_H^\kappa$, $\mathbb{F}$-stopping time τ, and $\mathcal{F}_\tau$-measurable random variable $\xi \in \mathbb{L}^2(\mathbb{P})$, let

$$(y^{\mathbb{P}}, z^{\mathbb{P}}, k^{\mathbb{P},+}, k^{\mathbb{P},-}) := (y^{\mathbb{P}}(\tau,\xi), z^{\mathbb{P}}(\tau,\xi), k^{\mathbb{P},+}(\tau,\xi), k^{\mathbb{P},-}(\tau,\xi)),$$

denote the unique solution to the following standard DRBSDE with upper obstacle S and lower obstacle L (existence and uniqueness have been proved under these assumptions in [26] among others)

$$\begin{cases} y_t^{\mathbb{P}} = \xi + \int_t^\tau \widehat{F}_s(y_s^{\mathbb{P}}, z_s^{\mathbb{P}})ds - \int_t^\tau z_s^{\mathbb{P}} dB_s + k_\tau^{\mathbb{P},-} - k_t^{\mathbb{P},-} - k_\tau^{\mathbb{P},+} + k_t^{\mathbb{P},+},\ 0 \leq t \leq \tau,\ \mathbb{P} - a.s. \\ L_t \leq y_t^{\mathbb{P}} \leq S_t,\ \mathbb{P} - a.s. \\ \int_0^t \left(y_{s^-}^{\mathbb{P}} - L_{s^-}\right) dk_s^{\mathbb{P},-} = \int_0^t \left(S_{s^-} - y_{s^-}^{\mathbb{P}}\right) dk_s^{\mathbb{P},+} = 0,\ \mathbb{P} - a.s.,\ \forall t \in [0,T]. \end{cases} \tag{4.2.6}$$

Remark 4.2.7. *Notice that the assumption that $L_t < S_t$ and $L_{t^-} < S_{t^-}$ implies that the non-decreasing processes $k^{\mathbb{P},+}$ and $k^{\mathbb{P},-}$ never act at the same time. This will be important later. This hypothesis is already present in [42] and [43].*

Everything is now ready for the

Definition 4.2.8. *We say $(Y,Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$ is a solution to the 2DRBSDE (4.2.5) if*

- $Y_T = \xi$, $\mathcal{P}_H^\kappa - q.s.$
- $\forall \mathbb{P} \in \mathcal{P}_H^\kappa$, *the process $V^{\mathbb{P}}$ defined below has paths of bounded variation $\mathbb{P} - a.s.$*

$$V_t^{\mathbb{P}} := Y_0 - Y_t - \int_0^t \widehat{F}_s(Y_s, Z_s)ds + \int_0^t Z_s dB_s, \ 0 \leq t \leq T, \ \mathbb{P} - a.s. \tag{4.2.7}$$

- *We have the following minimum condition for $0 \leq t \leq T$*

$$V_t^{\mathbb{P}} + k_t^{\mathbb{P},+} - k_t^{\mathbb{P},-} = \underset{\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}_t^{\mathbb{P}'} \left[V_T^{\mathbb{P}'} + k_T^{\mathbb{P}',+} - k_T^{\mathbb{P}',-} \right], \ \mathbb{P} - a.s., \ \forall \mathbb{P} \in \mathcal{P}_H^\kappa. \tag{4.2.8}$$

- $L_t \leq Y_t \leq S_t$, $\mathcal{P}_H^\kappa - q.s.$

Moreover, if there exists an aggregator for the family $(V^{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}_H^\kappa}$, that is to say a progressively measurable process V such that for all $\mathbb{P} \in \mathcal{P}_H^\kappa$,

$$V_t = V_t^{\mathbb{P}}, \ t \in [0,T], \ \mathbb{P} - a.s.,$$

then we say that (Y,Z,V) is a solution to the 2DRBSDE (4.2.5).

Remark 4.2.9. *The Definition 4.2.8 differs from the rest of the 2BSDE literature. Indeed, unlike in [89] for instance, the process $V^{\mathbb{P}}$ that we add in the definition of the 2BSDE is no longer non-decreasing, but is only assumed to have finite variation. This is mainly due to two competing effects. On the one hand, exactly as with standard RBSDEs with an upper obstacle, a non-increasing process has to be added to the solution in order to maintain it below the upper obstacle. But in the 2BSDE framework, another non-decreasing process also has to be added in order to push the process Y to stay above all the $y^{\mathbb{P}}$ (as is shown by the representation formula proved below in Theorem 4.3.1). This emphasizes once more that in the second-order framework, which is fundamentally non-linear, there is no longer any symmetry between a reflected 2BSDE with an upper or a lower obstacle. Notice that this was to be expected, since 2BSDEs are a natural generalization of the G-expectation introduced by Peng [78], which is an example of sublinear (and thus non-linear) expectation. We also would like to refer the reader to the recent paper by Pham and Zhang [81], whose problematics are strongly connected to ours. They study some norm estimates for semimartingales in the context of linear and sublinear expectations, and point out that there is a fundamental difference between non-linear submartingales and supermartingales (see their section 4.3). Translated in our framework, and using the intuition from the classical RBSDE theory, when the generator is equal to 0, a 2RBSDE with a lower obstacle should be a non-linear supermartingale, while a 2RBSDE with an upper obstacle should be a non-linear submartingale. In this sense, our results are a first step in the direction of the conjecture in section 4.3 of [81].*

4.2.5 DRBSDEs as a special case of 2DRBSDEs

In this subsection, we show how we can recover the usual theory. If H is linear in γ, that is to say

$$H_t(y,z,\gamma) := \frac{1}{2}\mathrm{Tr}\left[a_t^0\gamma\right] - f_t(y,z),$$

where $a^0 : [0,T]\times\Omega \to \mathbb{S}_d^{>0}$ is $\mathbb{F}$-progressively measurable and has uniform upper and lower bounds, then as in [89], we no longer need to assume any uniform continuity in ω in this case. Besides, the domain of F is restricted to a^0 and we have

$$\widehat{F}_t(y,z) = f_t(y,z).$$

If we further assume that there exists some $\mathbb{P}\in\overline{\mathcal{P}}_S$ such that $\widehat{a}$ and a^0 coincide $\mathbb{P}-a.s.$ and $\mathbb{E}^{\mathbb{P}}\left[\int_0^T |f_t(0,0)|^2\,dt\right] < +\infty$, then $\mathcal{P}_H^\kappa = \{\mathbb{P}\}$.

Then, we know that $V^{\mathbb{P}} + k_t^{\mathbb{P},+} - k_t^{\mathbb{P},-}$ is a $\mathbb{P}$-martingale with finite variation. Since $\mathbb{P}$ satisfy the martingale representation property, this martingale is also continuous, and therefore it is null. Thus we have

$$0 = k_t^{\mathbb{P},+} - k_t^{\mathbb{P},-} + V^{\mathbb{P}},\ \mathbb{P}-a.s.,$$

and the 2DRBSDE is equivalent to a standard DRBSDE. In particular, we see that $V^{\mathbb{P}}$ now becomes a finite variation process which decreases only when $Y_{t^-} = S_{t^-}$ and increases only when $Y_{t^-} = L_{t^-}$. This implies that $V^{\mathbb{P}}$ satisfies the usual Skorohod conditions. We would like to emphasize this fact here, since it will be useful later on to have a deeper understanding of the structure of the processes $\{V^{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}_H^\kappa}$.

4.3 Uniqueness, estimates and representations

4.3.1 A representation inspired by stochastic control

We have similarly as in Theorem 4.4 of [89]

Theorem 4.3.1. *Let Assumption **(D)** hold. Assume $\xi\in\mathbb{L}_H^{2,\kappa}$ and that (Y,Z) is a solution to the 2DRBSDE (4.2.5). Then, for any $\mathbb{P}\in\mathcal{P}_H^\kappa$ and $0\leq t_1<t_2\leq T$,*

$$Y_{t_1} = \underset{\mathbb{P}'\in\mathcal{P}_H^\kappa(t_1^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ y_{t_1}^{\mathbb{P}'}(t_2,Y_{t_2}),\ \mathbb{P}-a.s. \tag{4.3.9}$$

Consequently, the 2DRBSDE (4.2.5) has at most one solution in $\mathbb{D}_H^{2,\kappa}\times\mathbb{H}_H^{2,\kappa}$.

Proof. The proof is exactly the same as the proof of Theorem 3.1 in [66], so we will only sketch it. First, from the minimal condition (4.2.8), we deduce that for any $\mathbb{P}\in\mathcal{P}_H^\kappa$, the process $V^{\mathbb{P}} + k^{\mathbb{P},+} - k^{\mathbb{P},-}$ is a $\mathbb{P}$-submartingale. By the Doob-Meyer decomposition and the martingale representation property, its martingale part is continuous with finite variation and therefore null. Hence $V^{\mathbb{P}} + k^{\mathbb{P},+} - k^{\mathbb{P},-}$ is a non-decreasing process. Then, the inequality

$$Y_{t_1} \geq \underset{\mathbb{P}'\in\mathcal{P}_H^\kappa(t_1^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ y_{t_1}^{\mathbb{P}'}(t_2,Y_{t_2}),\ \mathbb{P}-a.s.,$$

is a simple consequence of a classical comparison Theorem. The reverse inequality is then obtained by standard linearization techniques using the Lipschitz properties of F, see [66] for the details. □

Remark 4.3.2. *Let us now justify the minimum condition* (4.2.8). *Assume for the sake of clarity that the generator* $\widehat{F}$ *is equal to* 0. *By the above Theorem, we know that if there exists a solution to the* 2*DRBSDE* (4.2.5), *then the process* Y *has to satisfy the representation* (4.3.9). *Therefore, we have a natural candidate for a possible solution of the* 2*DRBSDE. Now, assume that we could construct such a process* Y *satisfying the representation* (4.3.9) *and which has the decomposition* (4.2.5). *Then, taking conditional expectations in* $Y - y^{\mathbb{P}}$, *we end up with exactly the minimum condition* (4.2.8).

Finally, the following comparison Theorem follows easily from the classical one for DRBSDEs (see for instance [61]) and the representation (4.3.9).

Theorem 4.3.3. *Let* (Y, Z) *and* (Y', Z') (*resp.* $(y^{\mathbb{P}}, z^{\mathbb{P}}, k^{+,\mathbb{P}}, k^{-,\mathbb{P}})$ *and* $(y'^{\mathbb{P}}, z'^{\mathbb{P}}, k'^{+,\mathbb{P}}, k'^{-,\mathbb{P}})$) *be the solutions of the* 2*DRBSDEs* (*resp. DRBSDEs*) *with terminal conditions* ξ *and* ξ', *upper obstacles* S *and* S', *lower obstacles* L *and* L' *and generators* $\widehat{F}$ *and* $\widehat{F}'$ *respectively. Assume that they both verify Assumptions* ***(D)***, *that* $\mathcal{P}_H^\kappa \subset \mathcal{P}_{H'}^\kappa$ *and that we have* $\mathcal{P}_H^\kappa - q.s.$

$$\xi \leq \xi', \ \widehat{F}_t(y_t'^{\mathbb{P}}, z_t'^{\mathbb{P}}) \leq \widehat{F}_t'(y_t'^{\mathbb{P}}, z_t'^{\mathbb{P}}), \ L_t \leq L_t' \text{ and } S_t \geq S_t'.$$

Then $Y \leq Y'$, $\mathcal{P}_H^\kappa - q.s.$

Remark 4.3.4. *Unlike in the classical framework, even if the upper obstacles* S *and* S' *and the lower obstacles* L *and* L' *are identical, we cannot compare the processes* $V^{\mathbb{P}}$ *and* $V'^{\mathbb{P}}$. *This is due to the fact that these processes are not assumed to satisfy a Skorohod-type condition. This point was already mentioned in [66].*

4.3.2 A priori estimates

We will now try to obtain a priori estimates for the 2DRBSDEs. We emphasize immediately that the fact that the process $V^{\mathbb{P}}$ are only of finite variation makes the task a lot more difficult than in [66]. Indeed, we are now in a case which shares some similarities with standard doubly reflected BSDEs for which it is known that a priori estimates cannot be obtained without some regularity assumptions on the obstacles (for instance if one of them is a semimartingale). We assumed here that S was a semimartingale, a property which will be at the heart of our proofs. Nonetheless, even before this, we need to understand the fine structure of the processes $V^{\mathbb{P}}$. This is the object of the following proposition.

Proposition 4.3.5. *Let Assumption* ***(D)*** *hold. Assume* $\xi \in \mathbb{L}_H^{2,\kappa}$ *and* $(Y, Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$ *is a solution to the 2DRBSDE* (4.2.5). *Let* $\left\{(y^{\mathbb{P}}, z^{\mathbb{P}}, k^{+,\mathbb{P}}, k^{-,\mathbb{P}})\right\}_{\mathbb{P}\in\mathcal{P}_H^\kappa}$ *be the solutions of the corresponding DRBSDEs* (4.2.6). *Then we have the following results for all* $t \in [0, T]$ *and for all* $\mathbb{P} \in \mathcal{P}_H^\kappa$

(i) $V_t^{\mathbb{P},+} := \int_0^t \mathbf{1}_{y_{s^-}^{\mathbb{P}} < S_{s^-}} dV_s^{\mathbb{P}}$ *is a non-decreasing process,* $\mathbb{P} - a.s.$

(ii) $V_t^{\mathbb{P},-} := \int_0^t \mathbf{1}_{y_{s^-}^{\mathbb{P}} = S_{s^-}} dV_s^{\mathbb{P}} = -k_t^{\mathbb{P},+}$, $\mathbb{P} - a.s.$, *and is therefore a non-increasing process.*

Proof.

Let us fix a given $\mathbb{P} \in \mathcal{P}_H^\kappa$.

(i) Let τ_1 and τ_2 be two $\mathbb{F}$-stopping times such that for all $t \in [\tau_1, \tau_2)$, $y_{t^-}^{\mathbb{P}} < S_{t^-}$, $\mathbb{P} - a.s.$

Then, we know from the usual Skorohod condition that the process $k^{\mathbb{P},+}$ does not increase between τ_1 and τ_2. Now, we remind the reader that we showed in the proof of Theorem 4.3.1, that the process $V^{\mathbb{P}} + k^{+,\mathbb{P}} - k^{-,\mathbb{P}}$ is always non-decreasing. This necessarily implies that $V^{\mathbb{P}}$ must be non-decreasing between τ_1 and τ_2. Hence the first result.

(ii) Let now τ_1 and τ_2 be two $\mathbb{F}$-stopping times such that for all $t \in [\tau_1, \tau_2)$, $y^{\mathbb{P}}_{t^-} = S_{t^-}$, $\mathbb{P} - a.s.$

First, since the two obstacles are separated, we necessarily have $y^{\mathbb{P}}_{t^-} > L_{t^-}$, $\mathbb{P} - a.s.$ for every $t \in [\tau_1, \tau_2)$, which in turn implies that $k^{\mathbb{P},-}$ does not increase. Next, by the representation formula (4.3.9), we necessarily have $Y_{t^-} \geq y^{\mathbb{P}}_{t^-}$, $\mathbb{P} - a.s.$ for all t. Moreover, since we also have $Y_t \leq S_t$ by definition, this implies, since all the processes here are càdlàg, that we must have

$$Y_{t^-} = y^{\mathbb{P}}_{t^-} = S_{t^-},\ t \in [\tau_1, \tau_2),\ \mathbb{P} - a.s.$$

Using the fact that Y and $y^{\mathbb{P}}$ solve respectively a 2BSDE and a BSDE, we also have $\mathbb{P} - a.s.$

$$S_{t^-} + \Delta Y_t = Y_t = Y_u + \int_t^u \widehat{F}_s(Y_s, Z_s)ds - \int_t^u Z_s dB_s + V^{\mathbb{P}}_u - V^{\mathbb{P}}_t,\ \tau_1 \leq t \leq u < \tau_2$$
$$S_{t^-} + \Delta y^{\mathbb{P}}_t = y^{\mathbb{P}}_t = y^{\mathbb{P}}_u + \int_t^u \widehat{F}_s(y^{\mathbb{P}}_s, z^{\mathbb{P}}_s)ds - \int_t^u z^{\mathbb{P}}_s dB_s - k^{\mathbb{P},+}_u + k^{\mathbb{P},+}_t,\ \tau_1 \leq t \leq u < \tau_2.$$

Identifying the martingale parts above, we obtain that $Z_s = z^{\mathbb{P}}_s$, $ds \times \mathbb{P} - a.e.$ Then, identifying the finite variation parts, we have

$$Y_u - \Delta Y_t + \int_t^u \widehat{F}_s(Y_s, Z_s)ds + V^{\mathbb{P}}_u - V^{\mathbb{P}}_t = y^{\mathbb{P}}_u - \Delta y^{\mathbb{P}}_t + \int_t^u \widehat{F}_s(y^{\mathbb{P}}_s, z^{\mathbb{P}}_s)ds - k^{\mathbb{P},+}_u + k^{\mathbb{P},+}_t. \tag{4.3.10}$$

Now, we clearly have

$$\int_t^u \widehat{F}_s(Y_s, Z_s)ds = \int_t^u \widehat{F}_s(y^{\mathbb{P}}_s, z^{\mathbb{P}}_s)ds,$$

since $Z_s = z^{\mathbb{P}}_s$, $dt \times \mathbb{P} - a.e.$ and $Y_{s^-} = y^{\mathbb{P}}_{s^-} = S_{s^-}$ for all $s \in [t, u]$. Moreover, since $Y_{s^-} = y^{\mathbb{P}}_{s^-} = S_{s^-}$ for all $s \in [t, u]$ and since all the processes are càdlàg, the jumps of Y and $y^{\mathbb{P}}$ are equal to the jumps of S. Therefore, (4.3.10) can be rewritten

$$V^{\mathbb{P}}_u - V^{\mathbb{P}}_t = -k^{\mathbb{P},+}_u + k^{\mathbb{P},+}_t,$$

which is the desired result. □

The above Proposition is crucial for us. Indeed, we have actually shown that

$$V^{\mathbb{P}}_t = V^{\mathbb{P},+}_t - k^{\mathbb{P},+}_t,\ \mathbb{P} - a.s.,$$

where $V^{\mathbb{P},+}$ and $k^{\mathbb{P},+}$ are two non-decreasing processes which never act at the same time. Hence, we have obtained a Jordan decomposition for $V^{\mathbb{P}}$. Moreover, we can easily obtain a priori estimates for $k^{\mathbb{P},+}$ by using the fact that it is part of the solution of the DRBSDE (4.2.6). Notice that these estimates hold only because we assumed that the corresponding upper obstacle S was a semimartingale. This is this decomposition which will allow us to obtain the estimates.

Remark 4.3.6. *The above result is enough for us to obtain the desired a priori estimates. However, we can go further into the structure of the bounded variation processes* $V^{\mathbb{P}}$. *Indeed, arguing as in Proposition* 3.2 *in [66], we could also show that*

$$\mathbf{1}_{Y_{t^-} = L_{t^-}} dV^{\mathbb{P}}_t = \mathbf{1}_{Y_{t^-} = L_{t^-}} dk^{\mathbb{P},-}_t.$$

Notice however that we, a priori, cannot say anything about $V^{\mathbb{P}}$ when $L_{t^-} = y^{\mathbb{P}}_{t^-} < Y_{t^-}$, even though we showed that it could be known explicitly when $S_{t^-} = y^{\mathbb{P}}_{t^-}$. This emphasizes once more the fact that the upper and the lower obstacle in our context do not play a symmetric role.

We can now prove the following theorem.

Theorem 4.3.7. *Let Assumptions **(D)**, **(E)** and **(F)** hold. Assume $\xi \in \mathbb{L}^{2,\kappa}_H$ and $(Y,Z) \in \mathbb{D}^{2,\kappa}_H \times \mathbb{H}^{2,\kappa}_H$ is a solution to the 2DRBSDE (4.2.5). Let $\{(y^{\mathbb{P}}, z^{\mathbb{P}}, k^{\mathbb{P},+}, k^{\mathbb{P},-})\}_{\mathbb{P}\in\mathcal{P}^\kappa_H}$ be the solutions of the corresponding DRBSDEs (4.2.6). Then, there exists a constant C_κ depending only on κ, T and the Lipschitz constant of $\widehat{F}$ such that*

$$\|Y\|^2_{\mathbb{D}^{2,\kappa}_H} + \|Z\|^2_{\mathbb{H}^{2,\kappa}_H} + \sup_{\mathbb{P}\in\mathcal{P}^\kappa_H}\left\{\left\|y^{\mathbb{P}}\right\|^2_{\mathbb{D}^2(\mathbb{P})} + \left\|z^{\mathbb{P}}\right\|^2_{\mathbb{H}^2(\mathbb{P})} + \mathbb{E}^{\mathbb{P}}\left[\mathrm{Var}_{0,T}\left(V^{\mathbb{P}}\right)^2 + \left(k^{\mathbb{P},+}_T\right)^2 + \left(k^{\mathbb{P},-}_T\right)^2\right]\right\}$$
$$\leq C_\kappa\left(\|\xi\|^2_{\mathbb{L}^{2,\kappa}_H} + \phi^{2,\kappa}_H + \psi^{2,\kappa}_H + \varphi^{2,\kappa}_H + \zeta^{2,\kappa}_H\right)$$

Proof. First of all, since we assumed that S was a semimartingale, we can argue as in [26] to obtain that

$$dk^{\mathbb{P},+}_t \leq \widehat{F}^+_t(S_t, P_t)dt + dA^{\mathbb{P},+}_t \leq C\left(\left|\widehat{F}^0_t\right| + |S_t| + \left|\widehat{a}^{1/2}_t P_t\right|\right)dt + dA^{\mathbb{P},+}_t.$$

Hence,

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}_t\left[\left(k^{\mathbb{P},+}_T\right)^\kappa\right] &\leq C_\kappa \mathbb{E}^{\mathbb{P}}_t\left[\int_t^T \left|\widehat{F}^0_s\right|^\kappa ds + \left(\int_t^T |\widehat{a}^{\frac{1}{2}}_s P_s|^2 ds\right)^{\kappa/2} + \sup_{t\leq s\leq T}|S_s|^\kappa + \left(A^{\mathbb{P},+}_T\right)^\kappa\right]\\ &\leq C_\kappa\left(\left(\zeta^{2,\kappa}_H\right)^{1/2} + \mathbb{E}^{\mathbb{P}}_t\left[\int_t^T \left|\widehat{F}^0_s\right|^\kappa ds + \sup_{t\leq s\leq T}|S_s|^\kappa\right]\right).\end{aligned}$$

Let us now define

$$\tilde{\xi} := \xi - k^{\mathbb{P},+}_T,\ \tilde{y}^{\mathbb{P}} = y^{\mathbb{P}} - k^{\mathbb{P},+},\ \tilde{F}_t(y,z) := \widehat{F}_t\left(y + k^{\mathbb{P},+}_t, z\right).$$

Then, it is easy to see that $(\tilde{y}^{\mathbb{P}}, z^{\mathbb{P}}, k^{\mathbb{P},+})$ is the solution of the lower reflected BSDE with terminal condition $\tilde{\xi}$, generator $\tilde{F}$ and obstacle $L - k^{\mathbb{P},+}$. We can then once again apply Lemma 2 in [47] to obtain that there exists a constant C_κ depending only on κ, T and the Lipschitz constant of $\widehat{F}$, such that for all $\mathbb{P}$

$$\begin{aligned}\left|y^{\mathbb{P}}_t\right| &\leq C_\kappa \mathbb{E}^{\mathbb{P}}_t\left[\left|\tilde{\xi}\right|^\kappa + \int_t^T \left|\tilde{F}^0_s\right|^\kappa ds + \sup_{t\leq s\leq T}\left(\left(L_s - k^{\mathbb{P},+}_s\right)^+\right)^\kappa\right]\\ &\leq C_\kappa \mathbb{E}^{\mathbb{P}}_t\left[|\xi|^\kappa + \int_t^T \left|\widehat{F}^0_s\right|^\kappa ds + \sup_{t\leq s\leq T}(L^+_s)^\kappa + \left(k^{\mathbb{P},+}_T\right)^\kappa\right]\\ &\leq C_\kappa\left(\left(\zeta^{2,\kappa}_H\right)^{1/2} + \mathbb{E}^{\mathbb{P}}_t\left[|\xi|^\kappa + \int_t^T \left|\widehat{F}^0_s\right|^\kappa ds + \sup_{t\leq s\leq T}|S_s|^\kappa + \sup_{t\leq s\leq T}\left(L^+_s\right)^\kappa\right]\right).\end{aligned} \tag{4.3.11}$$

This immediately provides the estimate for $y^{\mathbb{P}}$. Now by definition of the norms, we obtain from (4.3.11) and the representation formula (4.3.9) that

$$\|Y\|^2_{\mathbb{D}^{2,\kappa}_H} \leq C_\kappa\left(\|\xi\|^2_{\mathbb{L}^{2,\kappa}_H} + \phi^{2,\kappa}_H + \psi^{2,\kappa}_H + \varphi^{2,\kappa}_H + \zeta^{2,\kappa}_H\right). \tag{4.3.12}$$

Now apply Itô's formula to $\left|y^{\mathbb{P}}\right|^2$ under $\mathbb{P}$. We get as usual for every $\varepsilon > 0$

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|\widehat{a}_t^{1/2} z_t^{\mathbb{P}}\right|^2 dt\right] &\leq C\mathbb{E}^{\mathbb{P}}\left[|\xi|^2 + \int_0^T \left|y_t^{\mathbb{P}}\right| \left(\left|\widehat{F}_t^0\right| + \left|y_t^{\mathbb{P}}\right| + \left|\widehat{a}_t^{1/2} z_t^{\mathbb{P}}\right|\right) dt\right] \\ &\quad + \mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|y_t^{\mathbb{P}}\right| d\left(k_t^{\mathbb{P},+} + k_t^{\mathbb{P},-}\right)\right] \\ &\leq C\left(\|\xi\|_{\mathbb{L}_H^{2,\kappa}} + \mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}\left|y_t^{\mathbb{P}}\right|^2 + \left(k_T^{\mathbb{P},-}\right)^2 + \left(\int_0^T \left|\widehat{F}_t^0\right| dt\right)^2\right]\right) \\ &\quad + \varepsilon\mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|\widehat{a}_t^{1/2} z_t^{\mathbb{P}}\right|^2 dt + \left|k_T^{\mathbb{P},+}\right|^2\right] + \frac{C^2}{\varepsilon}\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}\left|y_t^{\mathbb{P}}\right|^2\right]. \qquad (4.3.13)\end{aligned}$$

Then by definition of the DRBSDE (4.2.6), we easily have

$$\mathbb{E}^{\mathbb{P}}\left[\left|k_T^{\mathbb{P},+}\right|^2\right] \leq C_0\mathbb{E}^{\mathbb{P}}\left[|\xi|^2 + \sup_{0\leq t\leq T}\left|y_t^{\mathbb{P}}\right|^2 + \left(k_T^{\mathbb{P},-}\right)^2 + \int_0^T \left|\widehat{a}_t^{\frac{1}{2}} z_t^{\mathbb{P}}\right|^2 dt + \left(\int_0^T \left|\widehat{F}_t^0\right| dt\right)^2\right], \qquad (4.3.14)$$

for some constant C_0, independent of ε. Now set $\varepsilon := (2(1+C_0))^{-1}$ and plug (4.3.14) in (4.3.13). We obtain from the estimates for $y^{\mathbb{P}}$ and $k^{\mathbb{P},-}$

$$\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \left\|z^{\mathbb{P}}\right\|_{\mathbb{H}^2(\mathbb{P})} \leq C\left(\|\xi\|^2_{\mathbb{L}_H^{2,\kappa}} + \phi_H^{2,\kappa} + \psi_H^{2,\kappa} + \varphi_H^{2,\kappa} + \zeta_H^{2,\kappa}\right).$$

Then the estimate for $k^{\mathbb{P},+}$ comes from (4.3.14). Now that we have obtained the desired estimates for $y^{\mathbb{P}}$,$z^{\mathbb{P}}$, $k^{\mathbb{P},+}$, $k^{\mathbb{P},-}$ and Y, we can proceed further.

Exactly as above, we apply Itô's formula to $|Y|^2$ under each $\mathbb{P} \in \mathcal{P}_H^\kappa$. We have once more for every $\varepsilon > 0$ and using Proposition 4.3.5

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\int_0^T |\widehat{a}_t^{\frac{1}{2}} Z_t|^2 dt\right] &\leq C\mathbb{E}^{\mathbb{P}}\left[|\xi|^2 + \int_0^T |Y_t| \left(\left|\widehat{F}_t^0\right| + |Y_t| + |\widehat{a}_t^{\frac{1}{2}} Z_t|\right) dt\right] \\ &\quad + \mathbb{E}^{\mathbb{P}}\left[\int_0^T Y_t dV_t^{\mathbb{P},+} - \int_0^T Y_t dk_t^{\mathbb{P},+}\right] \\ &\leq C\left(\|\xi\|_{\mathbb{L}_H^{2,\kappa}} + \mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}|Y_t|^2 + \left(\int_0^T \left|\widehat{F}_t^0\right| dt\right)^2\right]\right) \\ &\quad + \varepsilon\mathbb{E}^{\mathbb{P}}\left[\int_0^T |\widehat{a}_t^{\frac{1}{2}} Z_t|^2 dt + \left|k_T^{\mathbb{P},+}\right|^2 + \left|V_T^{\mathbb{P},+}\right|^2 + \frac{C^2}{\varepsilon}\sup_{0\leq t\leq T}|Y_t|^2\right]. \qquad (4.3.15)\end{aligned}$$

Then by definition of our 2DRBSDE, we easily have

$$\mathbb{E}^{\mathbb{P}}\left[\left|V_T^{\mathbb{P},+}\right|^2\right] \leq C_0\mathbb{E}^{\mathbb{P}}\left[|\xi|^2 + \sup_{0\leq t\leq T}|Y_t|^2 + \int_0^T |\widehat{a}_t^{\frac{1}{2}} Z_t|^2 dt + \left|k_T^{\mathbb{P},+}\right|^2 + \left(\int_0^T |\widehat{F}_t^0| dt\right)^2\right], \qquad (4.3.16)$$

for some constant C_0, independent of ε.

Now set $\varepsilon := (2(1 + C_0))^{-1}$ and plug (4.3.16) in (4.3.15). One then gets

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|\widehat{a}_t^{1/2} Z_t\right|^2 dt\right] \leq C\mathbb{E}^{\mathbb{P}}\left[|\xi|^2 + \sup_{0\leq t\leq T}|Y_t|^2 + \left|k_T^{\mathbb{P},+}\right|^2 + \left(\int_0^T \left|\widehat{F}_t^0\right| dt\right)^2\right].$$

From this and the estimates for Y and $k^{\mathbb{P},+}$, we immediately obtain

$$\|Z\|_{\mathbb{H}_H^{2,\kappa}} \leq C\left(\|\xi\|^2_{\mathbb{L}_H^{2,\kappa}} + \phi_H^{2,\kappa} + \psi_H^{2,\kappa} + \varphi_H^{2,\kappa} + \zeta_H^{2,\kappa}\right).$$

Moreover, we deduce from (4.3.16) that

$$\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\left(V_T^{\mathbb{P},+}\right)^2\right] \leq C\left(\|\xi\|^2_{\mathbb{L}_H^{2,\kappa}} + \phi_H^{2,\kappa} + \psi_H^{2,\kappa} + \varphi_H^{2,\kappa} + \zeta_H^{2,\kappa}\right). \tag{4.3.17}$$

Finally, we have by definition of the total variation and the fact that the processes $V^{\mathbb{P},+}$ and $k^{\mathbb{P},+}$ are non-decreasing

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\operatorname*{Var}_{0,T}\left(V^{\mathbb{P}}\right)^2\right] &\leq C\mathbb{E}^{\mathbb{P}}\left[\operatorname*{Var}_{0,T}\left(V^{\mathbb{P},+}\right)^2 + \operatorname*{Var}_{0,T}\left(k^{\mathbb{P},+}\right)^2\right] \\
&= C\mathbb{E}^{\mathbb{P}}\left[\left(V_T^{\mathbb{P},+}\right)^2 + \left(k_T^{\mathbb{P},+}\right)^2\right] \\
&\leq C\left(\|\xi\|^2_{\mathbb{L}_H^{2,\kappa}} + \phi_H^{2,\kappa} + \psi_H^{2,\kappa} + \varphi_H^{2,\kappa} + \zeta_H^{2,\kappa}\right),
\end{aligned}$$

where we used the estimate for $k^{\mathbb{P},+}$ and (4.3.17) for the last inequality. □

Theorem 4.3.8. *Let Assumptions **(D)**, **(E)** and **(F)** hold. For $i = 1, 2$, let (Y^i, Z^i) be the solutions to the 2DRBSDE (4.2.5) with terminal condition ξ^i, upper obstacle S and lower obstacle L. Then, there exists a constant C_κ depending only on κ, T and the Lipschitz constant of F such that*

$$\begin{aligned}
&\left\|Y^1 - Y^2\right\|_{\mathbb{D}_H^{2,\kappa}} \leq C\left\|\xi^1 - \xi^2\right\|_{\mathbb{L}_H^{2,\kappa}} \\
&\left\|Z^1 - Z^2\right\|^2_{\mathbb{H}_H^{2,\kappa}} + \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}\left|V_t^{\mathbb{P},+,1} - V_t^{\mathbb{P},+,2}\right|^2 + \sup_{0\leq t\leq T}\left|V_t^{\mathbb{P},-,1} - V_t^{\mathbb{P},-,2}\right|^2\right] \\
&\leq C\left\|\xi^1 - \xi^2\right\|_{\mathbb{L}_H^{2,\kappa}}\left(\left\|\xi^1\right\|_{\mathbb{L}_H^{2,\kappa}} + \left\|\xi^1\right\|_{\mathbb{L}_H^{2,\kappa}} + (\phi_H^{2,\kappa})^{1/2} + (\psi_H^{2,\kappa})^{1/2} + (\varphi_H^{2,\kappa})^{1/2} + (\zeta_H^{2,\kappa})^{1/2}\right).
\end{aligned}$$

Remark 4.3.9. *We emphasize that in Theorem 4.3.8, we control the norm of both $V_t^{\mathbb{P},+,1} - V_t^{\mathbb{P},+,2}$ and $V_t^{\mathbb{P},-,1} - V_t^{\mathbb{P},-,2}$. This is crucial in our main existence Theorem 4.4.5.*

Proof. As in the previous Proposition, we can follow the proof of Lemma 3 in [47], to obtain that there exists a constant C_κ depending only on κ, T and the Lipschitz constant of F, such that for all $\mathbb{P}$

$$\left|y_t^{\mathbb{P},1} - y_t^{\mathbb{P},2}\right| \leq C_\kappa \mathbb{E}_t^{\mathbb{P}}\left[\left|\xi^1 - \xi^2\right|^\kappa\right]. \tag{4.3.18}$$

Now by definition of the norms, we get from (4.3.18) and the representation formula (4.3.9) that

$$\left\|Y^1 - Y^2\right\|^2_{\mathbb{D}_H^{2,\kappa}} \leq C_\kappa\left\|\xi^1 - \xi^2\right\|^2_{\mathbb{L}_H^{2,\kappa}}. \tag{4.3.19}$$

Next, the estimate for $V_t^{\mathbb{P},-,1} - V_t^{\mathbb{P},-,2}$ is immediate from the usual estimates for DRBSDEs (see for instance Theorem 3.2 in [26]), since we actually have from Proposition 4.3.5

$$V_t^{\mathbb{P},-,1} - V_t^{\mathbb{P},-,2} = k^{\mathbb{P},+,2} - k_t^{\mathbb{P},+,1}.$$

Applying Itô's formula to $\left|Y^1 - Y^2\right|^2$, under each $\mathbb{P} \in \mathcal{P}_H^\kappa$, leads to

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|\widehat{a}_t^{\frac{1}{2}}(Z_t^1 - Z_t^2)\right|^2 dt\right] &\leq C\mathbb{E}^{\mathbb{P}}\left[\left|\xi^1 - \xi^2\right|^2\right] + \mathbb{E}^{\mathbb{P}}\left[\int_0^T (Y_t^1 - Y_t^2) d(V_t^{\mathbb{P},1} - V_t^{\mathbb{P},2})\right] \\
&\quad + C\mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|Y_t^1 - Y_t^2\right| \left(\left|Y_t^1 - Y_t^2\right| + |\widehat{a}_t^{\frac{1}{2}}(Z_t^1 - Z_t^2)|\right) dt\right] \\
&\leq C\left(\left\|\xi^1 - \xi^2\right\|^2_{\mathbb{L}_H^{2,\kappa}} + \left\|Y^1 - Y^2\right\|^2_{\mathbb{D}_H^{2,\kappa}}\right) \\
&\quad + \frac{1}{2}\mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|\widehat{a}_t^{1/2}(Z_t^1 - Z_t^2)\right|^2 dt\right] \\
&\quad + C\left\|Y^1 - Y^2\right\|_{\mathbb{D}_H^{2,\kappa}} \left(\mathbb{E}^{\mathbb{P}}\left[\sum_{i=1}^2 \mathrm{Var}_{0,\mathrm{T}}\left(\mathrm{V}^{\mathbb{P},\mathrm{i}}\right)^2\right]\right)^{\frac{1}{2}}.
\end{aligned}$$

The estimate for $(Z^1 - Z^2)$ is now obvious from the above inequality and the estimates of Proposition 4.3.7. Finally, we have by definition for any $t \in [0, T]$

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}\left|V_t^{\mathbb{P},+,1} - V_t^{\mathbb{P},+,2}\right|^2\right] &\leq \mathbb{E}^{\mathbb{P}}\left[\left|\xi^1 - \xi^2\right|^2 + \sup_{0\leq t\leq T}\left\{\left|Y_t^1 - Y_t^2\right|^2 + \left|\int_t^T (Z_s^1 - Z_s^2) dB_s\right|^2\right\}\right] \\
&\quad + \mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|\widehat{F}_s(Y_s^1, Z_s^1) - \widehat{F}_s(Y_s^2, Z_s^2)\right|^2 ds + \sup_{0\leq t\leq T}\left|V_t^{\mathbb{P},-,1} - V_t^{\mathbb{P},-,2}\right|^2\right] \\
&\leq C\left(\left\|\xi^1 - \xi^2\right\|^2_{\mathbb{L}_H^{2,\kappa}} + \left\|Y^1 - Y^2\right\|^2_{\mathbb{D}_H^{2,\kappa}} + \left\|Z^1 - Z^2\right\|^2_{\mathbb{H}_H^{2,\kappa}}\right),
\end{aligned}$$

where we used the BDG inequality for the last step.

By all the previous estimates, this finishes the proof. □

4.3.3 Some properties of the solution

Now that we have proved the representation (4.3.9) and the a priori estimates of Theorems 4.3.7 and 4.3.8, we can show, as in the classical framework, that the solution Y of the 2DRBSDE is linked to some kind of Dynkin game. We emphasize that such a connection with games was already conjectured in [66]. After that, Bayraktar and Yao [11] showed in a purely Markovian context, that the value function of stochastic zero-sum differential game could be linked to the notion of 2DRBSDEs, even though these objects were not precisely defined in the paper (see their section 5.2). For any $t \in [0, T]$, denote $\mathcal{T}_{t,T}$ the set of $\mathbb{F}$-stopping times taking values in $[t, T]$.

Proposition 4.3.10. *Let (Y, Z) be the solution to the above 2DRBSDE (4.2.5). For any $(\tau, \sigma) \in \mathcal{T}_{0,T}$, define*

$$R_\tau^\sigma := S_\tau 1_{\tau<\sigma} + L_\sigma 1_{\sigma\leq\tau,\sigma<T} + \xi 1_{\tau\wedge\sigma=T}.$$

Then for each $t \in [0, T]$, for all $\mathbb{P} \in \mathcal{P}_H^\kappa$, we have $\mathbb{P} - a.s.$

$$\begin{aligned}
Y_t &= \operatorname*{ess\,sup^{\mathbb{P}}}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})} \operatorname*{ess\,inf}_{\tau\in\mathcal{T}_{t,T}} \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_{t,T}} \mathbb{E}_t^{\mathbb{P}'}\left[\int_t^{\tau\wedge\sigma} \widehat{F}_s(y_s^{\mathbb{P}'}, z_s^{\mathbb{P}'}) ds + R_\tau^\sigma\right] \\
&= \operatorname*{ess\,sup^{\mathbb{P}}}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})} \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_{t,T}} \operatorname*{ess\,inf}_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}_t^{\mathbb{P}'}\left[\int_t^{\tau\wedge\sigma} \widehat{F}_s(y_s^{\mathbb{P}'}, z_s^{\mathbb{P}'}) ds + R_\tau^\sigma\right].
\end{aligned}$$

Moreover, for any $\gamma \in [0,1]$, we have $\mathbb{P}-a.s.$

$$Y_t = \operatorname*{ess\,inf}_{\tau\in\mathcal{T}_{t,T}} \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_{t,T}} \mathbb{E}_t^{\mathbb{P}}\left[\int_t^{\tau\wedge\sigma} \widehat{F}_s(Y_s,Z_s)ds + K^{\mathbb{P},\gamma}_{\tau\wedge\sigma} - K^{\mathbb{P},\gamma}_t + R^\sigma_\tau\right]$$
$$= \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_{t,T}} \operatorname*{ess\,inf}_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}_t^{\mathbb{P}}\left[\int_t^{\tau\wedge\sigma} \widehat{F}_s(Y_s,Z_s)ds + K^{\mathbb{P},\gamma}_{\tau\wedge\sigma} - K^{\mathbb{P},\gamma}_t + R^\sigma_\tau\right],$$

where

$$K_t^{\mathbb{P},\gamma} := \gamma\int_0^t \mathbf{1}_{y^{\mathbb{P}}_{s^-}<S_{s^-}} dV_s^{\mathbb{P}} + (1-\gamma)\int_0^t \mathbf{1}_{Y_{s^-}>L_{s^-}} dV_s^{\mathbb{P}}.$$

Furthermore, for any $\mathbb{P}\in\mathcal{P}^\kappa_H$, the following stopping times are ε-optimal

$$\tau_t^{\varepsilon,\mathbb{P}} := \inf\left\{s\geq t,\ y^{\mathbb{P}}_s \geq S_s-\varepsilon,\ \mathbb{P}-a.s.\right\} \text{ and } \sigma_t^\varepsilon := \inf\left\{s\geq t,\ Y_s\leq L_s+\varepsilon,\ \mathcal{P}^\kappa_H - q.s.\right\}.$$

Remark 4.3.11. *Notice that the optimal stopping rules above are different in nature. Indeed, $\tau_t^{\varepsilon,\mathbb{P}}$ depends explicitly on the probability measures $\mathbb{P}$, because it depends on the process $y^{\mathbb{P}}$, while σ^ε_t only depends on Y. This situation shed once more light on the complete absence of symmetry between lower and upper obstacles in the second-order framework*

Remark 4.3.12. *The second result in the Proposition above may seem peculiar at first sight because of the degree of freedom introduced by the parameter γ. However, as shown in the proof below, we can find stopping times which are ε-optimizer for the corresponding stochastic game, and which roughly correspond (as expected) to the first hitting times of the obstacles. Since the latter are completely separated, we know from Proposition 4.3.5 that before hitting S,*

$$dV_t^{\mathbb{P}} = \mathbf{1}_{y^{\mathbb{P}}_{t^-}<S_{t^-}},$$

and that before hitting L,

$$dV_t^{\mathbb{P}} = \mathbf{1}_{Y_{t^-}>L_{t^-}}.$$

Thanks to this result, it is easy to see that we can change the value of γ as we want. In particular, if there is no upper obstacle, that is to say if $S=+\infty$, then taking $\gamma=0$, we recover the result of Proposition 3.1 in [66].

Proof. By Proposition 3.1 in [61], we know that for all $\mathbb{P}\in\mathcal{P}^\kappa_H$, $\mathbb{P}-a.s.$

$$y_t^{\mathbb{P}} = \operatorname*{ess\,inf}_{\tau\in\mathcal{T}_{t,T}} \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_{t,T}} \mathbb{E}_t^{\mathbb{P}}\left[\int_t^{\tau\wedge\sigma} \widehat{F}_s(y^{\mathbb{P}}_s,z^{\mathbb{P}}_s)ds + R^\sigma_\tau\right] = \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_{t,T}} \operatorname*{ess\,inf}_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}_t^{\mathbb{P}}\left[\int_t^{\tau\wedge\sigma} \widehat{F}_s(y^{\mathbb{P}}_s,z^{\mathbb{P}}_s)ds + R^\sigma_\tau\right].$$

Then the first equality is a simple consequence of the representation formula (4.3.9). For the second one, we proceed exactly as in the proof of Proposition 3.1 in [61]. Fix some $\mathbb{P}\in\mathcal{P}^\kappa_H$ and some $t\in[0,T]$ and some $\varepsilon>0$. It is then easy to show that for any $s\in[t,\tau_t^{\varepsilon,\mathbb{P}}]$, we have $y^{\mathbb{P}}_{s^-}<S_{s^-}$. In particular this implies that

$$dV_s^{\mathbb{P}} = \mathbf{1}_{y^{\mathbb{P}}_{s^-}<S_{s^-}} dV_s^{\mathbb{P}},\ s\in[t,\tau_t^{\varepsilon,\mathbb{P}}].$$

Let now $\sigma\in\mathcal{T}_{t,T}$. On the set $\{\tau_t^{\varepsilon,\mathbb{P}}<\sigma\}$, we have

$$\int_t^{\sigma\wedge\tau_t^{\varepsilon,\mathbb{P}}} \widehat{F}_s(Y_s,Z_s)ds + R^\sigma_{\tau_t^{\varepsilon,\mathbb{P}}} = \int_t^{\tau_t^{\varepsilon,\mathbb{P}}} \widehat{F}_s(Y_s,Z_s)ds + S_{\tau_t^{\varepsilon,\mathbb{P}}} \leq \int_t^{\tau_t^{\varepsilon,\mathbb{P}}} \widehat{F}_s(Y_s,Z_s)ds + y^{\mathbb{P}}_{\tau_t^{\varepsilon,\mathbb{P}}} + \varepsilon$$
$$\leq \int_t^{\tau_t^{\varepsilon,\mathbb{P}}} \widehat{F}_s(Y_s,Z_s)ds + Y_{\tau_t^{\varepsilon,\mathbb{P}}} + \varepsilon = Y_t + \int_t^{\tau_t^{\varepsilon,\mathbb{P}}} Z_s dB_s - \int_t^{\tau_t^{\varepsilon,\mathbb{P}}} \mathbf{1}_{y^{\mathbb{P}}_{s^-}<S_{s^-}} dV_s^{\mathbb{P}} + \varepsilon.$$

Then, notice that the process $(\mathbf{1}_{y^{\mathbb{P}}_{s^-}<S_{s^-}} - \mathbf{1}_{Y_{s^-}>L_{s^-}})dV^{\mathbb{P}}_s$ is non-decreasing. Therefore, we deduce

$$\begin{aligned}\int_t^{\sigma\wedge\tau_t^{\varepsilon,\mathbb{P}}} \widehat{F}_s(Y_s,Z_s)ds + R^{\sigma}_{\tau_t^{\varepsilon,\mathbb{P}}} &\leq Y_t + \int_t^{\tau_t^{\varepsilon,\mathbb{P}}} Z_s dB_s - \int_t^{\tau_t^{\varepsilon,\mathbb{P}}} \mathbf{1}_{y^{\mathbb{P}}_{s^-}<S_{s^-}} dV^{\mathbb{P}}_s \\ &\quad + (1-\gamma)\int_t^{\tau_t^{\varepsilon,\mathbb{P}}} \left(\mathbf{1}_{y^{\mathbb{P}}_{s^-}<S_{s^-}} - \mathbf{1}_{Y_{s^-}>L_{s^-}}\right) dV^{\mathbb{P}}_s + \varepsilon \\ &= Y_t + \int_t^{\tau_t^{\varepsilon,\mathbb{P}}} Z_s dB_s - (K^{\gamma}_{\tau_t^{\mathbb{P},\varepsilon}} - K^{\gamma}_t) + \varepsilon.\end{aligned}$$

Similarly on the set $\{\tau_t^{\varepsilon,\mathbb{P}} \geq \sigma\}$, we have

$$\begin{aligned}&\int_t^{\sigma\wedge\tau_t^{\varepsilon,\mathbb{P}}} \widehat{F}_s(Y_s,Z_s)ds + R^{\sigma}_{\tau_t^{\varepsilon,\mathbb{P}}} \leq \int_t^{\sigma} \widehat{F}_s(Y_s,Z_s)ds + \xi\mathbf{1}_{\sigma=T} + Y_\sigma \mathbf{1}_{\sigma<T} \\ &= Y_t + \int_t^{\sigma} Z_s dB_s - \int_t^{\sigma} \mathbf{1}_{y^{\mathbb{P}}_{s^-}<S_{s^-}} dV^{\mathbb{P}}_s \leq Y_t + \int_t^{\sigma} Z_s dB_s - (K^{\gamma}_{\sigma} - K^{\gamma}_t).\end{aligned}$$

With these two inequalities, we therefore have

$$\mathbb{E}^{\mathbb{P}}_t\left[\int_t^{\sigma\wedge\tau_t^{\varepsilon,\mathbb{P}}} \widehat{F}_s(Y_s,Z_s)ds + R^{\sigma}_{\tau_t^{\varepsilon,\mathbb{P}}} + K^{\gamma}_{\sigma\wedge\tau_t^{\mathbb{P},\varepsilon}} - K^{\gamma}_t\right] - \varepsilon \leq Y_t,\ \mathbb{P}-a.s. \tag{4.3.20}$$

We can prove similarly that for any $\tau \in \mathcal{T}_{t,T}$

$$\mathbb{E}^{\mathbb{P}}_t\left[\int_t^{\sigma_t^{\varepsilon,\mathbb{P}}\wedge\tau} \widehat{F}_s(Y_s,Z_s)ds + R^{\sigma_t^{\varepsilon,\mathbb{P}}}_{\tau} + K^{\gamma}_{\sigma_t^{\mathbb{P},\varepsilon}\wedge\tau} - K^{\gamma}_t\right] + \varepsilon \geq Y_t,\ \mathbb{P}-a.s. \tag{4.3.21}$$

Then, we can use Lemma 5.3 of [61] to finish the proof. □

Then, if we have more information on the obstacle S and its decomposition (4.2.3), we can give a more explicit representation for the processes $V^{\mathbb{P}}$, just as in the classical case (see Proposition 4.2 in [35]).

Assumptions (G):
S is a semi-martingale of the form

$$S_t = S_0 + \int_0^t U_s ds + \int_0^t P_s dB_s + C_t,\ \mathcal{P}^{\kappa}_H - q.s.$$

where C is càdlàg process of integrable variation such that the measure dC_t is singular with respect to the Lebesgue measure dt and which admits the following decomposition $C_t = C^+_t - C^-_t$, where C^+ and C^- are non-decreasing processes. Besides, U and V are respectively $\mathbb{R}$ and $\mathbb{R}^d$-valued $\mathcal{F}_t$ progressively measurable processes such that

$$\int_0^T (|U_t| + |P_t|^2)dt + C^+_T + C^-_T \leq +\infty,\ \mathcal{P}^{\kappa}_H - q.s.$$

Proposition 4.3.13. *Let Assumptions **(D)**, **(E)**, **(G)** and **(F)** hold. Let (Y, Z) be the solution to the 2DRBSDE (4.2.5), then for all $\mathbb{P} \in \mathcal{P}_H^\kappa$*

$$Z_t = P_t, \ dt \times \mathbb{P} - a.s. \text{ on the set } \{Y_{t^-} = S_{t^-}\}, \tag{4.3.22}$$

and there exists a progressively measurable process $(\alpha_t^{\mathbb{P}})_{0 \leq t \leq T}$ such that $0 \leq \alpha \leq 1$ and

$$-\mathbf{1}_{y_{t^-}^{\mathbb{P}} = S_{t^-}} dV_t^{\mathbb{P}} = \alpha_t^{\mathbb{P}} \mathbf{1}_{y_{t^-}^{\mathbb{P}} = S_{t^-}} \left(\left[\widehat{F}_t(S_t, P_t) + U_t \right]^+ dt + dC_t^+ \right).$$

Proof. First, for all $\mathbb{P} \in \mathcal{P}_H^\kappa$, the following holds $\mathbb{P} - a.s.$

$$S_t - Y_t = S_0 - Y_0 + \int_0^t \left(\widehat{F}_s(Y_s, Z_s) + U_s \right) ds - \int_0^t (Z_s - P_s) dB_s + V_t^{\mathbb{P}} + C_t^+ - C_t^-.$$

Now if we denote L_t the local time at 0 of $S_t - Y_t$, then by Itô-Tanaka formula under $\mathbb{P}$

$$\begin{aligned}(S_t - Y_t)^+ = (S_0 - Y_0)^+ + \int_0^t \mathbf{1}_{Y_{s^-} < S_{s^-}} \left(\widehat{F}_s(Y_s, Z_s) + U_s \right) ds - \int_0^t \mathbf{1}_{Y_{s^-} < S_{s^-}} (Z_s - P_s) dB_s \\ + \int_0^t \mathbf{1}_{Y_{s^-} < S_{s^-}} d(V_t^{\mathbb{P}} + C_t^+ - C_t^-) + \frac{1}{2} L_t \\ + \sum_{0 \leq s \leq t} (S_s - Y_s)^+ - (S_{s^-} - Y_{s^-})^+ - \mathbf{1}_{Y_{s^-} < S_{s^-}} \Delta(S_s - Y_s).\end{aligned}$$

However, we have $(S_t - Y_t)^+ = S_t - Y_t$, hence by identification of the martingale part

$$\mathbf{1}_{Y_{t^-} = S_{t^-}} (Z_t - P_t) dB_t = 0, \ \mathcal{P}_H^\kappa - q.s.,$$

from which the first statement is clear. Identifying the finite variation part, we obtain

$$\begin{aligned}&\mathbf{1}_{Y_{s^-} = S_{s^-}} \left(\widehat{F}_s(Y_s, Z_s) + U_s \right) ds + \mathbf{1}_{Y_{s^-} = S_{s^-}} d(V_s^{\mathbb{P}} + C_s^+ - C_s^-) \\ =& \frac{1}{2} L_s + \left((S_s - Y_s)^+ - (S_{s^-} - Y_{s^-})^+ - \mathbf{1}_{Y_{s^-} < S_{s^-}} \Delta(S_s - Y_s) \right).\end{aligned}$$

By Proposition 4.3.5, we know that $\mathbf{1}_{y_{s^-}^{\mathbb{P}} = S_{s^-}} dV_s^{\mathbb{P}}$ is a non-increasing process, while $\mathbf{1}_{y_{s^-}^{\mathbb{P}} < S_{s^-}} dV_s^{\mathbb{P}}$ is a non-decreasing process. Furthermore, we have

$$\mathbf{1}_{Y_{s^-} = S_{s^-}} dV_s^{\mathbb{P}} = \mathbf{1}_{y_{s^-}^{\mathbb{P}} = S_{s^-}} dV_s^{\mathbb{P}} + \mathbf{1}_{y_{s^-}^{\mathbb{P}} < Y_{s^-} = S_{s^-}} dV_s^{\mathbb{P}}.$$

Since we also know that the jump part, L and C^- are non-decreasing processes, we obtain

$$\begin{aligned}-\mathbf{1}_{y_{s^-}^{\mathbb{P}} = S_{s^-}} dV_s^{\mathbb{P}} \leq& \mathbf{1}_{y_{s^-}^{\mathbb{P}} = S_{s^-}} \left(\left(\widehat{F}_s(Y_s, Z_s) + U_s \right) ds + dC_s^+ \right) \\ &+ \mathbf{1}_{y_{s^-}^{\mathbb{P}} < Y_{s^-} = S_{s^-}} \left(\left(\widehat{F}_s(Y_s, Z_s) + U_s \right) ds + dC_s^+ + dV_s^{\mathbb{P}} \right).\end{aligned}$$

Since, $\mathbf{1}_{y_{s^-}^{\mathbb{P}} = S_{s^-}} dV_s^{\mathbb{P}}$ and $\mathbf{1}_{y_{s^-}^{\mathbb{P}} < Y_{s^-} = S_{s^-}} \left(\left(\widehat{F}_s(Y_s, Z_s) + U_s \right) ds + dC_s^+ + dV_s^{\mathbb{P}} \right)$ never act at the same time by definition, the second statement follows easily. □

Remark 4.3.14. *If we assume also that L is a semimartingale, then we can obtain exactly the same type of results as in Corollary 3.1 in [66], using the exact same arguments.*

4.4 A constructive proof of existence

We have shown in Theorem 4.3.1 that if a solution exists, it will necessarily verify the representation (4.2.8). This gives us a natural candidate for the solution as a supremum of solutions to standard DRBSDEs. However, since those DRBSDEs are all defined on the support of mutually singular probability measures, it seems difficult to define such a supremum, because of the problems raised by the negligible sets. In order to overcome this, Soner, Touzi and Zhang proposed in [89] a pathwise construction of the solution to a 2BSDE. Let us describe briefly their strategy.

The first step is to define pathwise the solution to a standard BSDE. For simplicity, let us consider first a BSDE with a generator equal to 0. Then, we know that the solution is given by the conditional expectation of the terminal condition. In order to define this solution pathwise, we can use the so-called regular conditional probability distribution (r.p.c.d. for short) of Stroock and Varadhan [91]. In the general case, the idea is similar and consists on defining BSDEs on a shifted canonical space.

Finally, we have to prove measurability and regularity of the candidate solution thus obtained, and the decomposition (4.2.5) is obtained through a non-linear Doob-Meyer decomposition. Our aim in this section is to extend this approach in the presence of obstacles. We emphasize that most of the proofs are now standard, and we will therefore only sketch them, insisting particularly on the new difficulties appearing in the present setting.

4.4.1 Shifted spaces

For the convenience of the reader, we recall below some of the notations introduced in [89].

• For $0 \leq t \leq T$, denote by $\Omega^t := \left\{\omega \in C\left([t,T],\mathbb{R}^d\right),\ w(t) = 0\right\}$ the shifted canonical space, B^t the shifted canonical process, $\mathbb{P}_0^t$ the shifted Wiener measure and $\mathbb{F}^t$ the filtration generated by B^t.

• For $0 \leq s \leq t \leq T$ and $\omega \in \Omega^s$, define the shifted path $\omega^t \in \Omega^t$

$$\omega_r^t := \omega_r - \omega_t,\ \forall r \in [t,T].$$

• For $0 \leq s \leq t \leq T$ and $\omega \in \Omega^s$, $\widetilde{\omega} \in \Omega^t$ define the concatenation path $\omega \otimes_t \widetilde{\omega} \in \Omega^s$ by

$$(\omega \otimes_t \widetilde{\omega})(r) := \omega_r 1_{[s,t)}(r) + (\omega_t + \widetilde{\omega}_r) 1_{[t,T]}(r),\ \forall r \in [s,T].$$

• For $0 \leq s \leq t \leq T$ and a $\mathcal{F}_T^s$-measurable random variable ξ on Ω^s, for each $\omega \in \Omega^s$, define the shifted $\mathcal{F}_T^t$-measurable random variable $\xi^{t,\omega}$ on Ω^t by

$$\xi^{t,\omega}(\widetilde{\omega}) := \xi(\omega \otimes_t \widetilde{\omega}),\ \forall \widetilde{\omega} \in \Omega^t.$$

Similarly, for an $\mathbb{F}^s$-progressively measurable process X on $[s,T]$ and $(t,\omega) \in [s,T] \times \Omega^s$, the shifted process $\left\{X_r^{t,\omega}, r \in [t,T]\right\}$ is $\mathbb{F}^t$-progressively measurable.

• For a $\mathbb{F}$-stopping time τ, the r.c.p.d of $\mathbb{P}$ (denoted $\mathbb{P}_\tau^\omega$) is a probability measure on $\mathcal{F}_T$ such that

$$\mathbb{E}_\tau^{\mathbb{P}}[\xi](\omega) = \mathbb{E}^{\mathbb{P}_\tau^\omega}[\xi],\ \text{for } \mathbb{P} - a.e.\ \omega.$$

It also induces naturally a probability measure $\mathbb{P}^{\tau,\omega}$ (that we also call the r.c.p.d. of $\mathbb{P}$) on $\mathcal{F}_T^{\tau(\omega)}$ which in particular satisfies that for every bounded and $\mathcal{F}_T$-measurable random variable ξ

$$\mathbb{E}^{\mathbb{P}^\omega_\tau}[\xi] = \mathbb{E}^{\mathbb{P}^{\tau,\omega}}\left[\xi^{\tau,\omega}\right].$$

• We define similarly as in Section 4.2 the set $\bar{\mathcal{P}}_S^t$, by restricting to the shifted canonical space Ω^t, and its subset $\mathcal{P}_H^t$.

• Finally, we define the "shifted" generator

$$\widehat{F}_s^{t,\omega}(\widetilde{\omega}, y, z) := F_s(\omega \otimes_t \widetilde{\omega}, y, z, \widehat{a}_s^t(\widetilde{\omega})),\ \forall (s, \widetilde{\omega}) \in [t, T] \times \Omega^t.$$

Notice that thanks to Lemma 4.1 in [90], this generator coincides for $\mathbb{P}$-a.e. ω with the shifted generator as defined above, that is to say $F_s(\omega\otimes_t\widetilde{\omega}, y, z, \widehat{a}_s(\omega\otimes_t\widetilde{\omega}))$. The advantage of the chosen "shifted" generator is that it inherits the uniform continuity in ω under the $\mathbb{L}^\infty$ norm of F.

4.4.2 A first existence result when ξ is in $\mathrm{UC_b}(\Omega)$

Let us define for all $\omega \in \Omega$, $\Lambda^*(\omega) := \sup_{0\leq s\leq t} \Lambda_s(\omega)$, where

$$\Lambda_t^2(\omega) := \sup_{\mathbb{P}\in\mathcal{P}_H^t} \mathbb{E}^{\mathbb{P}}\left[\left|\xi^{t,\omega}\right|^2 + \int_t^T |\widehat{F}_s^{t,\omega}(0,0)|^2 ds + \sup_{t\leq s\leq T}\left|S_s^{t,\omega}\right|^2 + \sup_{t\leq s\leq T}\left((L_s^{t,\omega})^+\right)^2\right].$$

By Assumption **(F)**, we can check directly that

$$\Lambda_t(\omega) < \infty \text{ for all } (t,\omega)\in[0,T]\times\Omega. \tag{4.4.23}$$

To prove existence, we define the following value process X_t pathwise

$$X_t(\omega) := \sup_{\mathbb{P}\in\mathcal{P}_H^t} \mathcal{Y}_t^{\mathbb{P},t,\omega}(T,\xi),\ \text{for all } (t,\omega)\in[0,T]\times\Omega, \tag{4.4.24}$$

where, for any $(t_1,\omega)\in[0,T]\times\Omega$, $\mathbb{P}\in\mathcal{P}_H^{t_1}, t_2\in[t_1,T]$, and any $\mathcal{F}_{t_2}$-measurable $\eta\in\mathbb{L}^2(\mathbb{P})$, we denote $\mathcal{Y}_{t_1}^{\mathbb{P},t_1,\omega}(t_2,\eta) := y_{t_1}^{\mathbb{P},t_1,\omega}$, where $\left(y^{\mathbb{P},t_1,\omega}, z^{\mathbb{P},t_1,\omega}, k^{\mathbb{P},+,t_1,\omega}, k^{\mathbb{P},-,t_1,\omega}\right)$ is the solution of the following DRBSDE with upper obstacle $S^{t_1,\omega}$ and lower obstacle $L^{t_1,\omega}$ on the shifted space Ω^{t_1} under $\mathbb{P}$

$$\begin{aligned} y_s^{\mathbb{P},t_1,\omega} &= \eta^{t_1,\omega} + \int_s^{t_2} \widehat{F}_r^{t_1,\omega}\left(y_r^{\mathbb{P},t_1,\omega}, z_r^{\mathbb{P},t_1,\omega}\right)dr - \int_s^{t_2} z_r^{\mathbb{P},t_1,\omega}dB_r^{t_1} - k_{t_2}^{\mathbb{P},+,t_1,\omega} + k_s^{\mathbb{P},+,t_1,\omega} \\ &\quad + k_{t_2}^{\mathbb{P},-,t_1,\omega} - k_s^{\mathbb{P},-,t_1,\omega},\ \mathbb{P}-a.s. \end{aligned} \tag{4.4.25}$$

$$\begin{aligned} &L_t^{t_1,\omega} \leq y_t^{\mathbb{P},t_1,\omega} \leq S_t^{t_1,\omega},\ \mathbb{P}-a.s. \\ &\int_{t_1}^{t_2}\left(S_{s^-}^{t_1,\omega} - y_{s^-}^{\mathbb{P},t_1,\omega}\right)dk_s^{\mathbb{P},+,t_1,\omega} = \int_{t_1}^{t_2}\left(y_{s^-}^{\mathbb{P},t_1,\omega} - L_{s^-}^{t_1,\omega}\right)dk_s^{\mathbb{P},-,t_1,\omega} = 0,\ \mathbb{P}-a.s. \end{aligned} \tag{4.4.26}$$

Notice that since we assumed that S was a $\mathbb{P}$-semimartingale for all $\mathbb{P}\in\mathcal{P}_H^\kappa$, then for all $(t,\omega)\in[0,T]\times\Omega$, $S^{t,\omega}$ is also a $\mathbb{P}$-semimartingale for all $\mathbb{P}\in\mathcal{P}_H^{t,\kappa}$. Furthermore, we have the decomposition

$$S_s^{t,\omega} = S_t^{t,\omega} + \int_t^s P_u^{t,\omega}dB_u^t + A_s^{\mathbb{P},t,\omega},\ \mathbb{P}-a.s.,\ \text{for all } \mathbb{P}\in\mathcal{P}_H^{t,\kappa}, \tag{4.4.27}$$

where $A^{\mathbb{P},t,\omega}$ is a bounded variation process under $\mathbb{P}$. Besides, we have by Assumption **(F)**

$$\zeta_H^{t,\omega} := \sup_{\mathbb{P}\in\mathcal{P}_H^{t,\kappa}} \left(\mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}_{t\le s\le T}^{\mathbb{P}} \mathbb{E}_t^{H,\mathbb{P}}\left[\int_t^T \left|(\widehat{a}_s^t)^{1/2}P_s^{t,\omega}\right|^2 ds + \left(A_T^{\mathbb{P},t,\omega,+}\right)^2\right]\right]\right)^{1/2} < +\infty.$$

In view of the Blumenthal zero-one law, $\mathcal{Y}_t^{\mathbb{P},t,\omega}(T,\xi)$ is constant for any given (t,ω) and $\mathbb{P}\in\mathcal{P}_H^t$. Let us now answer the question of measurability of the process X

Lemma 4.4.1. *Let Assumptions **(D)** and **(F)** hold and consider some ξ in $\mathrm{UC_b}(\Omega)$. Then for all $(t,\omega)\in[0,T]\times\Omega$ we have $|X_t(\omega)|\le C(1+\zeta_H^{t,\omega}+\Lambda_t(\omega))$. Moreover, for all $(t,\omega,\omega')\in[0,T]\times\Omega^2$, $|X_t(\omega)-X_t(\omega')|\le C\rho(\|\omega-\omega'\|_t)$. Consequently, V_t is $\mathcal{F}_t$-measurable for every $t\in[0,T]$.*

Proof. (i) For each $(t,\omega)\in[0,T]\times\Omega$, since $S^{t,\omega}$ is a semimartingale with decomposition (4.4.27), we know that we have

$$\begin{aligned} dk_s^{\mathbb{P},+,t,\omega} &\le \left(\widehat{F}_s^{t,\omega}(S_s^{t,\omega},P_s^{t,\omega})\right)^+ ds + dA_s^{\mathbb{P},t,\omega,+} \\ &\le C\left(\left|\widehat{F}_s^{t,\omega}(0)\right| + \left|S_s^{t,\omega}\right| + \left|(\widehat{a}_s^t)^{1/2}P_s^{t,\omega}\right|\right)ds + dA_s^{\mathbb{P},t,\omega,+}. \end{aligned}$$

Hence,

$$\mathbb{E}^{\mathbb{P}}\left[\left(k_T^{\mathbb{P},+,t,\omega}\right)^2\right] \le C\left(\zeta_H^{t,\omega} + \mathbb{E}^{\mathbb{P}}\left[\int_t^T \left|\widehat{F}_s^{t,\omega}(0)\right|^2 ds + \sup_{t\le s\le T}\left|S_s^{t,\omega}\right|^2\right]\right).$$

Let now α be some positive constant which will be fixed later and let $\eta\in(0,1)$. By Itô's formula we have using (4.4.26)

$$\begin{aligned} &e^{\alpha t}\left|y_t^{\mathbb{P},t,\omega}\right|^2 + \int_t^T e^{\alpha s}\left|(\widehat{a}_s^t)^{1/2}z_s^{\mathbb{P},t,\omega}\right|^2 ds \le e^{\alpha T}\left|\xi^{t,\omega}\right|^2 + 2C\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega}\right|\left|\widehat{F}_s^{t,\omega}(0)\right| ds \\ &\quad + 2C\int_t^T \left|y_s^{\mathbb{P},t,\omega}\right|\left(\left|y_s^{\mathbb{P},t,\omega}\right| + \left|(\widehat{a}_s^t)^{1/2}z_s^{\mathbb{P},t,\omega}\right|\right)ds - 2\int_t^T e^{\alpha s}y_{s^-}^{\mathbb{P},t,\omega}z_s^{\mathbb{P},t,\omega}dB_s^t \\ &\quad + 2\int_t^T e^{\alpha s}S_{s^-}^{t,\omega}dk_s^{\mathbb{P},+,t,\omega} - 2\int_t^T e^{\alpha s}L_{s^-}^{t,\omega}dk_s^{\mathbb{P},-,t,\omega} - \alpha\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega}\right|^2 ds \\ &\le e^{\alpha T}\left|\xi^{t,\omega}\right|^2 + \int_t^T e^{\alpha s}\left|\widehat{F}_s^{t,\omega}(0)\right|^2 ds - 2\int_t^T e^{\alpha s}y_{s^-}^{\mathbb{P},t,\omega}z_s^{\mathbb{P},t,\omega}dB_s^t + \eta\int_t^T e^{\alpha s}\left|(\widehat{a}_s^t)^{1/2}z_s^{\mathbb{P},n}\right|^2 ds \\ &\quad + \left(2C + C^2 + \frac{C^2}{\eta} - \alpha\right)\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega}\right|^2 ds + 2\sup_{t\le s\le T} e^{\alpha s}\left|S_s^{t,\omega}\right|\left(k_T^{\mathbb{P},+,t,\omega} - k_t^{\mathbb{P},+,t,\omega}\right) \\ &\quad + 2\sup_{t\le s\le T} e^{\alpha s}(L_s^{t,\omega})^+\left(k_T^{\mathbb{P},-,t,\omega} - k_t^{\mathbb{P},-,t,\omega}\right). \end{aligned}$$

Now choose α such that $\nu := \alpha - 2C - C^2 - \frac{C^2}{\eta} \ge 0$. We obtain for all $\varepsilon > 0$

$$\begin{aligned} &e^{\alpha t}\left|y_t^{\mathbb{P},t,\omega}\right|^2 + (1-\eta)\int_t^T e^{\alpha s}\left|(\widehat{a}_s^t)^{1/2}z_s^{\mathbb{P},t,\omega}\right|^2 ds \le e^{\alpha T}\left|\xi^{t,\omega}\right|^2 \\ &\quad + \int_t^T e^{\alpha s}\left|\widehat{F}_s^{t,\omega}(0,0)\right|^2 ds + \frac{1}{\varepsilon}\left(\sup_{t\le s\le T} e^{\alpha s}(L_s^{t,\omega})^+\right)^2 + \varepsilon\left(k_T^{\mathbb{P},-,t,\omega} - k_t^{\mathbb{P},-,t,\omega}\right)^2 \\ &\quad + \left(\sup_{t\le s\le T} e^{\alpha s}\left|S_s^{t,\omega}\right|\right)^2 + \left(k_T^{\mathbb{P},+,t,\omega} - k_t^{\mathbb{P},+,t,\omega}\right)^2 - 2\int_t^T e^{\alpha s}y_{s^-}^{\mathbb{P},t,\omega}z_s^{\mathbb{P},t,\omega}dB_s^t. \end{aligned} \tag{4.4.28}$$

Taking expectation and using (4.4.28) yields with η small enough

$$\left|y_t^{\mathbb{P},t,\omega}\right|^2+\mathbb{E}^{\mathbb{P}}\left[\int_t^T|(\widehat{a}_s^t)^{\frac{1}{2}}z_s^{\mathbb{P},t,\omega}|^2ds\right]\leq C\left(\Lambda_t^2(\omega)+\left(\zeta_H^{t,\omega}\right)^2\right)+\varepsilon\mathbb{E}^{\mathbb{P}}\left[(k_T^{\mathbb{P},-,t,\omega}-k_t^{\mathbb{P},-,t,\omega})^2\right].$$

Now by definition, we also have for some constant C_0 independent of ε

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[(k_T^{\mathbb{P},-,t,\omega}-k_t^{\mathbb{P},-,t,\omega})^2\right]\leq C_0\left(\Lambda_t^2(\omega)+\left(\zeta_H^{t,\omega}\right)^2+\mathbb{E}^{\mathbb{P}}\left[\int_t^T\left|y_s^{\mathbb{P},t,\omega}\right|^2ds\right]\right)\\+C_0\mathbb{E}^{\mathbb{P}}\left[\int_t^T\left|(\widehat{a}_s^t)^{1/2}z_s^{\mathbb{P},t,\omega}\right|^2ds\right].\end{aligned}$$

Choosing $\varepsilon=\frac{1}{2C_0}$, Gronwall inequality then implies $|y_t^{\mathbb{P},t,\omega}|^2\leq C(1+\Lambda_t(\omega))$. The result then follows by arbitrariness of $\mathbb{P}$.

(ii) The proof is exactly the same as above, except that one has to use uniform continuity in ω of $\xi^{t,\omega}$, $\widehat{F}^{t,\omega}$, $S^{t,\omega}$ and $L^{t,\omega}$. Indeed, for each $(t,\omega)\in[0,T]\times\Omega$ and $\mathbb{P}\in\mathcal{P}_H^{t,\kappa}$, let α be some positive constant which will be fixed later and let $\eta\in(0,1)$. By Itô's formula we have, since $\widehat{F}$ is uniformly Lipschitz

$$\begin{aligned}&e^{\alpha t}\left|y_t^{\mathbb{P},t,\omega}-y_t^{\mathbb{P},t,\omega'}\right|^2+\int_t^T e^{\alpha s}\left|(\widehat{a}_s^t)^{1/2}(z_s^{\mathbb{P},t,\omega}-z_s^{\mathbb{P},t,\omega'})\right|^2ds\leq e^{\alpha T}\left|\xi^{t,\omega}-\xi^{t,\omega'}\right|^2\\
&+2C\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega}-y_s^{\mathbb{P},t,\omega'}\right|\left(\left|y_s^{\mathbb{P},t,\omega}-y_s^{\mathbb{P},t,\omega'}\right|+\left|(\widehat{a}_s^t)^{\frac{1}{2}}(z_s^{\mathbb{P},t,\omega}-z_s^{\mathbb{P},t,\omega'})\right|\right)ds\\
&+2C\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega}-y_s^{\mathbb{P},t,\omega'}\right|\left|\widehat{F}_s^{t,\omega}(y_s^{\mathbb{P},t,\omega},z_s^{\mathbb{P},t,\omega})-\widehat{F}_s^{t,\omega'}(y_s^{\mathbb{P},t,\omega},z_s^{\mathbb{P},t,\omega})\right|ds\\
&+2\int_t^T e^{\alpha s}(y_{s^-}^{\mathbb{P},t,\omega}-y_{s^-}^{\mathbb{P},t,\omega'})d\left(k_s^{\mathbb{P},-,t,\omega}-k_s^{\mathbb{P},-,t,\omega'}-k_s^{\mathbb{P},+,t,\omega}+k_s^{\mathbb{P},+,t,\omega'}\right)\\
&-\alpha\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega}-y_s^{\mathbb{P},t,\omega'}\right|^2ds-2\int_t^T e^{\alpha s}(y_{s^-}^{\mathbb{P},t,\omega}-y_{s^-}^{\mathbb{P},t,\omega'})(z_s^{\mathbb{P},t,\omega}-z_s^{\mathbb{P},t,\omega'})dB_s^t\\
&\leq e^{\alpha T}\left|\xi^{t,\omega}-\xi^{t,\omega'}\right|^2+\int_t^T e^{\alpha s}\left|\widehat{F}_s^{t,\omega}(y_s^{\mathbb{P},t,\omega},z_s^{\mathbb{P},t,\omega})-\widehat{F}_s^{t,\omega'}(y_s^{\mathbb{P},t,\omega},z_s^{\mathbb{P},t,\omega})\right|^2ds\\
&+\left(2C+C^2+\tfrac{C^2}{\eta}-\alpha\right)\int_t^T e^{\alpha s}\left|y_s^{\mathbb{P},t,\omega}-y_s^{\mathbb{P},t,\omega'}\right|^2ds+\eta\int_t^T e^{\alpha s}\left|(\widehat{a}_s^t)^{\frac{1}{2}}(z_s^{\mathbb{P},t,\omega}-z_s^{\mathbb{P},t,\omega'})\right|^2ds\\
&-2\int_t^T e^{\alpha s}(y_{s^-}^{\mathbb{P},t,\omega}-y_{s^-}^{\mathbb{P},t,\omega'})(z_s^{\mathbb{P},t,\omega}-z_s^{\mathbb{P},t,\omega'})dB_s^t\\
&+2\int_t^T e^{\alpha s}(y_{s^-}^{\mathbb{P},t,\omega}-y_{s^-}^{\mathbb{P},t,\omega'})d\left(k_s^{\mathbb{P},-,t,\omega}-k_s^{\mathbb{P},-,t,\omega'}-k_s^{\mathbb{P},+,t,\omega}+k_s^{\mathbb{P},+,t,\omega'}\right).\end{aligned}$$

By the Skorohod condition (4.4.26), we also have

$$\begin{aligned}&\int_t^T e^{\alpha s}(y_{s^-}^{\mathbb{P},t,\omega}-y_{s^-}^{\mathbb{P},t,\omega'})d\left(k_s^{\mathbb{P},-,t,\omega}-k_s^{\mathbb{P},-,t,\omega'}-k_s^{\mathbb{P},+,t,\omega}+k_s^{\mathbb{P},+,t,\omega'}\right)\\
&\leq\int_t^T e^{\alpha s}(L_{s^-}^{t,\omega}-L_{s^-}^{t,\omega'})d(k_s^{\mathbb{P},-,t,\omega}-k_s^{\mathbb{P},-,t,\omega'})-\int_t^T e^{\alpha s}(S_{s^-}^{t,\omega}-S_{s^-}^{t,\omega'})d(k_s^{\mathbb{P},+,t,\omega}-k_s^{\mathbb{P},+,t,\omega'}).\end{aligned}$$

Now choose α such that $\nu := \alpha - 2C - C^2 - \frac{C^2}{\eta} \geq 0$. We obtain for all $\varepsilon > 0$

$$
\begin{aligned}
& e^{\alpha t} \left| y_t^{\mathbb{P},t,\omega} - y_t^{\mathbb{P},t,\omega'} \right|^2 + (1-\eta) \int_t^T e^{\alpha s} \left| (\widehat{a}_s^t)^{1/2} (z_s^{\mathbb{P},t,\omega} - z_s^{\mathbb{P},t,\omega'}) \right|^2 ds \\
& \leq e^{\alpha T} \left| \xi^{t,\omega} - \xi^{t,\omega'} \right|^2 + \int_t^T e^{\alpha s} \left| \widehat{F}_s^{t,\omega}(y_s^{\mathbb{P},t,\omega}, z_s^{\mathbb{P},t,\omega}) - \widehat{F}_s^{t,\omega'}(y_s^{\mathbb{P},t,\omega}, z_s^{\mathbb{P},t,\omega}) \right|^2 ds \\
& \quad + \frac{1}{\varepsilon} \left(\sup_{t \leq s \leq T} e^{\alpha s} (L_s^{t,\omega} - L_s^{t,\omega'})^+ \right)^2 + \varepsilon (k_T^{\mathbb{P},-,t,\omega} - k_T^{\mathbb{P},-,t,\omega'} - k_t^{\mathbb{P},-,t\omega} + k_t^{\mathbb{P},-,t,\omega'})^2 \\
& \quad + \frac{1}{\varepsilon} \left(\sup_{t \leq s \leq T} e^{\alpha s} \left| S_s^{t,\omega} - S_s^{t,\omega'} \right| \right)^2 + \varepsilon (k_T^{\mathbb{P},+,t,\omega} - k_T^{\mathbb{P},+,t,\omega'} - k_t^{\mathbb{P},+,t\omega} + k_t^{\mathbb{P},+,t,\omega'})^2 \\
& \quad - 2 \int_t^T e^{\alpha s} (y_{s^-}^{\mathbb{P},t,\omega} - y_{s^-}^{\mathbb{P},t,\omega'}) (z_s^{\mathbb{P},t,\omega} - z_s^{\mathbb{P},t,\omega'}) dB_s^t.
\end{aligned}
\tag{4.4.29}
$$

The end of the proof is then similar to the previous step, using the uniform continuity in ω of ξ, F and S. $\square$

Then, we show the same dynamic programming principle as Proposition 4.7 in [90] and Proposition 4.1 in [66]. The proof being exactly the same, we omit it.

Proposition 4.4.2. *Under Assumptions **(D)**, **(F)** and for $\xi \in \mathrm{UC_b}(\Omega)$, we have for all $0 \leq t_1 < t_2 \leq T$ and for all $\omega \in \Omega$*

$$
X_{t_1}(\omega) = \sup_{\mathbb{P} \in \mathcal{P}_H^{t_1,\kappa}} \mathcal{Y}_{t_1}^{\mathbb{P},t_1,\omega}(t_2, X_{t_2}^{t_1,\omega}).
$$

Define now for all (t,ω), the $\mathbb{F}^+$-progressively measurable process

$$
X_t^+ := \overline{\lim_{r \in \mathbb{Q} \cap (t,T], r \downarrow t}} X_r. \tag{4.4.30}
$$

We have the following result whose proof is the same as the one of Lemma 4.2 in [66].

Lemma 4.4.3. *Under the conditions of the previous Proposition, we have*

$$
X_t^+ = \lim_{r \in \mathbb{Q} \cap (t,T], r \downarrow t} X_r, \ \mathcal{P}_H^\kappa - q.s.
$$

and thus X^+ is càdlàg $\mathcal{P}_H^\kappa - q.s.$

Proceeding exactly as in Steps 1 et 2 of the proof of Theorem 4.5 in [90], we can then prove that X^+ is a strong doubly reflected $\widehat{F}$-supermartingale (in the sense of Definition 4.6.6 in the Appendix). Then, using the Doob-Meyer decomposition proved in the Appendix in Theorem 4.6.8 for all $\mathbb{P}$, we know that there exists a unique ($\mathbb{P} - a.s.$) process $\overline{Z}^{\mathbb{P}} \in \mathbb{H}^2(\mathbb{P})$ and unique non-decreasing càdlàg square integrable processes $A^{\mathbb{P}}$, $B^{\mathbb{P}}$ and $C^{\mathbb{P}}$ such that

- $X_t^+ = X_0^+ - \int_0^t \widehat{F}_s(X_s^+, \overline{Z}_s^{\mathbb{P}})ds + \int_0^t \overline{Z}_s^{\mathbb{P}} dB_s + B_t^{\mathbb{P}} - A_t^{\mathbb{P}} + C_t^{\mathbb{P}}, \ \mathbb{P} - a.s., \ \forall \mathbb{P} \in \mathcal{P}_H^\kappa.$
- $L_t \leq X_t^+ \leq S_t, \ \mathbb{P} - a.s., \ \forall \mathbb{P} \in \mathcal{P}_H^\kappa.$
- $\int_0^T \left(S_{t^-} - X_{t^-}^+ \right) dA_t^{\mathbb{P}} = \int_0^T \left(X_{t^-}^+ - L_{t^-} \right) dB_t^{\mathbb{P}} = 0, \ \mathbb{P} - a.s., \ \forall \mathbb{P} \in \mathcal{P}_H^\kappa.$

We then define $V^{\mathbb{P}} := A^{\mathbb{P}} - B^{\mathbb{P}} - C^{\mathbb{P}}$. By Karandikar [50], since X^+ is a càdlàg semimartingale, we can define a universal process $\overline{Z}$ which aggregates the family $\left\{\overline{Z}^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H^\kappa\right\}$.

We next prove the representation (4.3.9) for X and X^+.

Proposition 4.4.4. *Assume that $\xi \in UC_b(\Omega)$ and that Assumptions **(D)** and **(F)** hold. Then we have*

$$X_t = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t,\mathbb{P})} \mathcal{Y}_t^{\mathbb{P}'}(T,\xi) \text{ and } X_t^+ = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+,\mathbb{P})} \mathcal{Y}_t^{\mathbb{P}'}(T,\xi),\ \mathbb{P}-a.s.,\ \forall \mathbb{P} \in \mathcal{P}_H^\kappa.$$

Proof. The proof for the representations is the same as the proof of proposition 4.10 in [90], since we also have a stability result for RBSDEs under our assumptions. □

Finally, we have to check that the minimum condition (4.2.8) holds. However, this can be done exactly as in [66], so we refer the reader to that proof.

4.4.3 Main result

We are now in position to state the main result of this section

Theorem 4.4.5. *Let $\xi \in \mathcal{L}_H^{2,\kappa}$ and let Assumptions **(D)**, **(E)** and **(F)** hold. Then*
1) There exists a unique solution $(Y,Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$ of the 2DRBSDE (4.2.5).
2) Moreover, if in addition we choose to work under either of the following model of set theory (we refer the reader to [39] for more details)

(i) *Zermelo-Fraenkel set theory with axiom of choice (ZFC) plus the Continuum Hypothesis (CH).*

(ii) *ZFC plus the negation of CH plus Martin's axiom.*

Then there exists a unique solution $(Y,Z,V) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa} \times \mathbb{V}_H^{2,\kappa}$ of the 2DRBSDE (4.2.5).

Proof. The proof of the existence part follows the lines of the proof of Theorem 4.7 in [89], using the estimates of Theorem 4.3.8, so we only insist on the points which do not come directly from the proofs mentioned above. The idea is to approximate the terminal condition ξ by a sequence $(\xi^n)_{n\geq 0} \subset \mathrm{UC_b}(\Omega)$. Then, we use the estimates of Theorem 4.3.8 to pass to the limit as in the proof of Theorem 4.6 in [89]. The main point in this context is that for each n, if we consider the Jordan decomposition of $V^{\mathbb{P},n}$ into the non-decreasing process $V^{\mathbb{P},+,n}$ and the non-increasing process $V^{\mathbb{P},-,n}$, then the estimates of Theorem 4.3.8 ensure that these processes converge to some $V^{\mathbb{P},+}$ and $V^{\mathbb{P},-}$, which are respectively non-decreasing and non-increasing. Hence we are sure that the limit $V^{\mathbb{P}}$ has indeed bounded variation.

Concerning the fact that we can aggregate the family $\left(V^{\mathbb{P}}\right)_{\mathbb{P}\in\mathcal{P}_H^\kappa}$, it can be deduced as follows. First, if $\xi \in \mathrm{UC_b}(\Omega)$, we know, using the same notations as above that the solution verifies

$$X_t^+ = X_0^+ - \int_0^t \widehat{F}_s(X_s^+, \overline{Z}_s)ds + \int_0^t \overline{Z}_s dB_s - K_t^{\mathbb{P}},\ \mathbb{P}-a.s.,\ \forall \mathbb{P} \in \mathcal{P}_H^\kappa.$$

Now, we know from (4.4.30) that X^+ is defined pathwise, and so is the Lebesgue integral

$$\int_0^t \widehat{F}_s(X_s^+, \overline{Z}_s)ds.$$

In order to give a pathwise definition of the stochastic integral, we would like to use the recent results of Nutz [68]. However, the proof in this paper relies on the notion of medial limits, which may or may not exist depending on the model of set theory chosen. They exists in the model (i) above, which is the one considered by Nutz, but we know from [39] (see statement 22O(l) page 55) that they also do in the model (ii). Therefore, provided we work under either one of these models, the stochastic integral $\int_0^t \overline{Z}_s dB_s$ can also be defined pathwise. We can therefore define pathwise

$$V_t := X_0^+ - X_t^+ - \int_0^t \widehat{F}_s(X_s^+, \overline{Z}_s)ds + \int_0^t \overline{Z}_s dB_s,$$

and V is an aggregator for the family $\left(V^{\mathbb{P}}\right)_{\mathbb{P}\in\mathcal{P}_H^\kappa}$, that is to say that it coincides $\mathbb{P}-a.s.$ with $V^{\mathbb{P}}$, for every $\mathbb{P}\in\mathcal{P}_H^\kappa$.

In the general case when $\xi\in\mathcal{L}_H^{2,\kappa}$, the family is still aggregated when we pass to the limit. □

Remark 4.4.6. *For more discussions on the axioms of the set theory considered here, we refer the reader to the Remark* 4.2 *in [66].*

4.5 Applications: Israeli options and Dynkin games

4.5.1 Game options

We first recall the definition of an Israeli (or game) option, and we refer the reader to [51], [41] and the references therein for more details. An Israeli option is a contract between a broker (seller) and a trader (buyer). The specificity is that both can decide to exercise before the maturity date T. If the trader exercises first at a time t then the broker pays him the (random) amount L_t. If the broker exercises before the trader at time t, the trader will be given from him the quantity $S_t \geq L_t$, and the difference $S_t - L_t$ is as to be understood as a penalty imposed on the seller for canceling the contract. In the case where they exercise simultaneously at t, the trader payoff is L_t and if they both wait till the maturity of the contract T, the trader receives the amount ξ. In other words, this is an American option which has the specificity that the seller can also "exercise" early. This therefore is a typical Dynkin game. We assume throughout this section that the processes L and S satisfy Assumptions **(F)** and **(G)**.

To sum everything up, if we consider that the broker exercises at a stopping time $\tau\leq T$ and the trader at another time $\sigma\leq T$ then the trader receive from the broker the following payoff:

$$H(\sigma,\tau) := S_\tau 1_{\tau<\sigma} + L_\sigma 1_{\sigma\leq\tau} + \xi 1_{\sigma\wedge\tau=T}$$

Before introducing volatility uncertainty, let us first briefly recall how the fair price and the hedging of such an option is related to DRBSDEs in a classical financial market. We fix a probability measure $\mathbb{P}$, and we assume that the market contains one riskless asset, whose price is assumed w.l.o.g. to be equal to 1, and one risky asset. We furthermore assume that if the broker adopts a strategy π (which is an adapted process in $\mathbb{H}^2(\mathbb{P})$ representing the percentage of his total wealth invested in the risky asset), then his wealth process has the following expression

$$X_t^{\mathbb{P}} = \xi + \int_t^T b(s, X_s^{\mathbb{P}}, \pi_s^{\mathbb{P}})ds - \int_t^T \pi_s^{\mathbb{P}}\sigma_s dW_s\,, \mathbb{P}-a.s.$$

where W is a Brownian motion under $\mathbb{P}$, b is convex and Lipschitz with respect to (x,π). We also suppose that the process $(b(t,0,0))_{t\leq T}$ is square-integrable and $(\sigma_t)_{t\leq T}$ is invertible and its inverse is bounded. It was then proved in [51] and [41] that the fair price and an hedging strategy for the Israeli option described above can be obtained through the solution of a DRBSDE. More precisely, we have

Theorem 4.5.1. *The fair price of the game option and the corresponding hedging strategy are given by the pair* $(y^{\mathbb{P}}, \pi^{\mathbb{P}}) \in \mathbb{D}^2(\mathbb{P}) \times \mathbb{H}^2(\mathbb{P})$ *solving the following DRBSDE*

$$\begin{cases} y_t^{\mathbb{P}} = \xi + \int_t^T b(s, y_s^{\mathbb{P}}, \pi_s^{\mathbb{P}})ds - \int_t^T \pi_s^{\mathbb{P}} \sigma_s dW_s + k_t^{\mathbb{P}} - k_t^{\mathbb{P}},\ \mathbb{P} - a.s. \\ L_t \leq y_t^{\mathbb{P}} \leq S_t,\ \mathbb{P} - a.s. \\ \int_0^T (y_{t^-}^{\mathbb{P}} - L_{t^-}) dk_t^{\mathbb{P},-} = \int_0^T (S_{t^-} - y_{t^-}^{\mathbb{P}}) dk_t^{\mathbb{P},+} = 0. \end{cases}$$

Moreover, for any $\varepsilon > 0$*, the following stopping times are* ε*-optimal after* t *for the seller and the buyer respectively*

$$D_t^{1,\varepsilon,\mathbb{P}} := \inf\left\{s \geq t,\ y_s^{\mathbb{P}} \geq S_s - \varepsilon\right\},\ D_t^{2,\varepsilon,\mathbb{P}} := \inf\left\{s \geq t,\ y_s^{\mathbb{P}} \leq L_s + \varepsilon\right\}.$$

Let us now extend this result to the uncertain volatility framework. We still consider a financial market with two assets and assume now that the wealth process has the following dynamic when the chosen strategy is π

$$X_t = \xi + \int_t^T b(s, X_s, \pi_s)ds - \int_t^T \pi_s dB_s,\ \mathcal{P}_H^\kappa - q.s.,$$

where B is the canonical process and b is assumed to satisfy Assumptions **(D)**, and where ξ belongs to $\mathcal{L}_H^{2,\kappa}$. Then, following the ideas of [66], it is natural to consider as a superhedging price for the option the quantity

$$Y_t = \underset{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ y_t^{\mathbb{P}'}.$$

Indeed, this amount is greater than the price at time t of the the same Israeli option under any probability measure. Hence, if the seller receives this amount, he should always be able to hedge his position. We emphasize however that we are not able to guarantee that this price is optimal in the sense that it is the lowest value for which we can find a super-replicating strategy. This interesting question is left for future research.

Symmetrically, if the seller charges less than the following quantity for the option at time t,

$$\widetilde{Y}_t := \underset{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}}\ y_t^{\mathbb{P}'},$$

then it will be clearly impossible for him to find a hedge. $\widetilde{Y}$ appears then as a subhedging price.

Hence, we have obtained a whole interval of prices, given by $[\widetilde{Y}_t, Y_t]$, which we can formally think as arbitrage free, even though a precise definition of this notion in an uncertain market is outside the scope of this paper. These two quantities can be linked to the notion of second-order 2DRBSDEs. Indeed, this is immediate for Y, and for $\widetilde{Y}$, we need to introduce a "symmetric" definition for the 2DRBSDEs.

Definition 4.5.2. *For* $\xi \in \mathbb{L}_H^{2,\kappa}$*, we consider the following type of equations satisfied by a pair of progressively-measurable processes* (Y, Z)

- $Y_T = \xi$, $\mathcal{P}_H^\kappa - q.s.$

- *$\forall \mathbb{P} \in \mathcal{P}_H^\kappa$, the process $V^\mathbb{P}$ defined below has paths of bounded variation $\mathbb{P}-a.s.$*

$$V_t^\mathbb{P} := Y_0 - Y_t - \int_0^t \widehat{F}_s(Y_s, Z_s)ds + \int_0^t Z_s dB_s,\ 0 \leq t \leq T,\ \mathbb{P}-a.s. \tag{4.5.31}$$

- *We have the following maximum condition for $0 \leq t \leq T$*

$$V_t^\mathbb{P} + k_t^{\mathbb{P},+} - k_t^{\mathbb{P},-} = \underset{\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,sup}^\mathbb{P}} \mathbb{E}_t^{\mathbb{P}'}\left[V_T^{\mathbb{P}'} + k_T^{\mathbb{P}',+} - k_T^{\mathbb{P}',-}\right],\ \mathbb{P}-a.s.,\ \forall \mathbb{P} \in \mathcal{P}_H^\kappa. \tag{4.5.32}$$

- *$L_t \leq Y_t \leq S_t$, $\mathcal{P}_H^\kappa - q.s.$*

This Definition is symmetric to Definition 4.2.8 in the sense that if (Y, Z) solves an equation as in Definition 4.5.2, then $(-Y, -Z)$ solves a 2DRBSDE (in the sense of Definition 4.2.8) with terminal condition $-\xi$, generator $\tilde{g}(y, z) := -g(-y, -z)$, lower obstacle $-S$ and upper obstacle $-L$. With this remark, it is clear that we can deduce a wellposedness theory for the above equations. In particular, we have the following representation

$$Y_t = \underset{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+,\mathbb{P})}{\operatorname{ess\,inf}} y_t^{\mathbb{P}'},\ \mathbb{P}-a.s.,\ \text{for any } \mathbb{P} \in \mathcal{P}_H^\kappa. \tag{4.5.33}$$

We then have the following result.

Theorem 4.5.3. *The superhedging and subhedging prices Y and $\widetilde{Y}$ are respectively the unique solution of the 2DRBSDE with terminal condition ξ, generator b, lower obstacle L, upper obstacle S in the sense of Definitions 4.2.8 and 4.5.2 respectively. The corresponding hedging strategies are then given by Z and $\widetilde{Z}$.*

Moreover, for any $\varepsilon > 0$ and for any $\mathbb{P}$, the following stopping times are ε-optimal after t for the seller and the buyer respectively

$$D_t^{1,\varepsilon,\mathbb{P}} := \inf\left\{s \geq t,\ y_s^\mathbb{P} \geq S_s - \varepsilon,\ \mathbb{P}-a.s.\right\},\ D_t^{2,\varepsilon} := \inf\left\{s \geq t,\ Y_s \leq L_s + \varepsilon,\ \mathcal{P}_H^\kappa - q.s.\right\}.$$

4.5.2 A first step towards Dynkin games under uncertainty

It is already known that doubly reflected BSDE are intimately connected to Dynkin games (see [27] for instance). More generally, since the seminal paper by Fleming and Souganidis [38], two person zero-sum stochastic differential games have been typically studied through two approaches. One uses the viscosity theory and aims at showing that the value function of the game is the unique viscosity solution of the associated Hamilton-Jacobi-Bellman-Isaacs equation (HJBI in short), while the other relates the value function to the solution of a BSDE. We are of course more interested in the second one. To name but a few of the contributions in the literature, Buckdahn and Li [20] defined precisely the value function of the game via BSDEs, while more recently Bayraktar and Yao used doubly reflected BSDEs. Before specializing the discussion to Dynkin games, we would like to refer the reader to the very recent work of Pham and Zhang [82], which studies a weak formulation of two person zero-sum game and points out several formal connections with the 2BSDE theory.

We naturally want to obtain the same kind of result with 2DRBSDEs, with an additional uncertainty component in the game, which will be induced by the fact that we are working simultaneously under a family of mutually singular probability measures. We will focus here on the construction of a game whose upper and lower values can be expressed as a solution of 2DRBSDE. We insist that we prove only that a given solution to a 2DRBSDE provides a solution to the corresponding Dynkin game described below. However, we are not able, as in [27], to construct the solution of the 2DRBSDE directly from the solution of the Dynkin game. Moreover, we also face a difficult technical problem related to Assumption **(H)**, which prevents our result to be comprehensive.

Let us now describe what we mean precisely by a Dynkin game with uncertainty. Two players P1 and P2 are facing each other in a game. A strategy of a player consists in picking a stopping time. Let us say that P1 chooses $\tau \in \mathcal{T}_{0,T}$ and P2 chooses $\sigma \in \mathcal{T}_{0,T}$. Then the game stipulates that P1 will pay to P2 the following random payoff

$$R_t(\tau,\sigma) := \int_t^{\tau\wedge\sigma} g_s ds + S_\tau 1_{\tau<\sigma} + L_\sigma 1_{\sigma\leq\tau,\sigma<T} + \xi 1_{\tau\wedge\sigma=T},$$

where g, S and L are $\mathbb{F}$-progressively measurable processes satisfying Assumptions **(D)**, **(G)** and **(F)**. In particular, the upper obstacle S is a semimartingale.

Then naturally, P1 will try to minimize the expected amount that he will have to pay, but taking into account the fact that both P2 and the "Nature" (which we interpret as a third player, represented by the uncertainty that the player has with respect to the underlying probability measure) can play against him. Symmetrically, P2 will try to maximize his expected returns, considering that both P1 and the Nature are antagonist players. This leads us to introduce the following upper and lower values of the robust Dynkin game

$$\overline{V}_t := \operatorname*{ess\,inf}_{\tau\in\mathcal{T}_{t,T}} \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_{t,T}} \operatorname*{ess\,sup^{\mathbb{P}}}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[R_t(\tau,\sigma)],\ \mathbb{P}-a.s.$$

$$\underline{V}_t := \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_{t,T}} \operatorname*{ess\,inf}_{\tau\in\mathcal{T}_{t,T}} \operatorname*{ess\,inf^{\mathbb{P}}}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[R_t(\tau,\sigma)],\ \mathbb{P}-a.s.$$

Remark 4.5.4. *In order to be completely rigorous, we should have made the dependence in $\mathbb{P}$ of the two functions above explicit, because it is not clear that an aggregator exists a priori. Nonetheless, we will prove in this section, that they both correspond to the solution of 2DRBSDE, and therefore that the aggregator indeed exists. Therefore, for the sake of clarity, we will always omit this dependence.*

$\overline{V}$ is the maximal amount that P1 will agree to pay in order to take part in the game. Symmetrically, $\underline{V}$ is the minimal amount that P2 must receive in order to accept to take part to the game. Unlike in the classical setting without uncertainty, for which there is only one value on which the 2 players can agree, in our context there is generally a whole interval of admissible values for the game. Indeed, we have the following easy result

Lemma 4.5.5. *We have for $t \in [0,T]$*

$$\overline{V}_t \geq \underline{V}_t,\ \mathcal{P}_H^\kappa - q.s.$$

Therefore the admissible values for the game are the interval $[\underline{V}_t, \overline{V}_t]$.

Proof. Let $t\in[0,T]$, $\mathbb{P}\in\mathcal{P}_H^\kappa$ and $\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})$. For any $(\tau,\sigma)\in\mathcal{T}_{t,T}\times\mathcal{T}_{t,T}$, we have clearly

$$\operatorname*{ess\,sup^{\mathbb{P}}}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[R_t(\tau,\sigma)] \geq \operatorname*{ess\,inf}_{\tau\in\mathcal{T}_{t,T}} \operatorname*{ess\,inf^{\mathbb{P}}}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[R_t(\tau,\sigma)],\ \mathbb{P}-a.s.$$

Then we can take the essential supremum with respect to σ on both sides of the inequality, and the result follows. $\square$

Now, in order to link the solution of the above robust Dynkin game to 2DRBSDEs, we will need to assume a min-max property which is closely related to the usual Isaacs condition for the classical Dynkin games. Given the length of the paper, we will not try to verify this assumption. Nonetheless, we emphasize that a related result was indeed proved in [65] in the context of a robust utility maximization problem. Even more in the spirit of our paper, Nutz and Zhang [69] also showed such a result (at least at time $t = 0$ and under sufficient regularity assumptions, see their Theorem 3.4) when there is only one player. We are convinced that their results could be generalized to our framework and leave this interesting problem to future research.

Assumption (H):

We suppose that the following "min-max" property are satisfied. For any $\mathbb{P} \in \mathcal{P}_H^\kappa$

$$\operatorname*{ess\,inf}_{\tau\in\mathcal{T}_{t,T}} \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_{t,T}} \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[R_t(\tau,\sigma)] = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})} \operatorname*{ess\,inf}_{\tau\in\mathcal{T}_{t,T}} \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_{t,T}} \mathbb{E}_t^{\mathbb{P}'}[R_t(\tau,\sigma)], \ \mathbb{P}-a.s. \tag{4.5.34}$$

$$\operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_{t,T}} \operatorname*{ess\,inf}_{\tau\in\mathcal{T}_{t,T}} \operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[R_t(\tau,\sigma)] = \operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})} \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_{t,T}} \operatorname*{ess\,inf}_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}_t^{\mathbb{P}'}[R_t(\tau,\sigma)], \ \mathbb{P}-a.s. \tag{4.5.35}$$

It is clear from Proposition 4.3.10 that the right-hand side of (4.5.34) can be expressed as the solution of 2DRBSDE with terminal condition ξ, generator g, lower obstacle L and upper obstacle S. We deduce immediately the following result.

Theorem 4.5.6. *Let Assumption **(H)** hold. Let (Y,Z) (resp. $(\widetilde{Y},\widetilde{Z})$) be a solution to the 2DRBSDE in the sense of Definition 4.2.8 (resp. in the sense of Definition 4.5.2) with terminal condition ξ, generator g, lower obstacle L and upper obstacle S. Then we have for any $t\in[0,T]$*

$$\overline{V}_t = Y_t, \ \mathcal{P}_H^\kappa - q.s.$$
$$\underline{V}_t = \widetilde{Y}_t, \ \mathcal{P}_H^\kappa - q.s.$$

Moreover, unless $\mathcal{P}_H^\kappa$ is reduced to a singleton, we have $\overline{V} > \underline{V}$, $\mathcal{P}_H^\kappa - q.s.$

Proof. The two equalities are obvious. Moreover, if for each $\mathbb{P}\in\mathcal{P}_H^\kappa$, we let $y^{\mathbb{P}}$ be the solution of the DRBSDE with terminal condition ξ, generator g, lower obstacle L and upper obstacle S, we have by (4.5.33)

$$\overline{V}_t = \operatorname*{ess\,sup}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})} y_t^{\mathbb{P}'}, \ \mathbb{P}-a.s., \text{ and } \underline{V}_t = \operatorname*{ess\,inf}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t^+,\mathbb{P})} y_t^{\mathbb{P}'}, \ \mathbb{P}-a.s.,$$

which implies the last result. $\square$

4.6 Appendix: Doubly reflected g-supersolution and martingales

In this section, we extend some of the results of [77] and [66] concerning g-supersolution of BSDEs and RBSDEs to the case of DRBSDEs. Let us note that many of the results below are obtained using similar ideas as in [77] and [66], but we still provide most of them since, to the best of our knowledge, they do not appear anywhere else in the literature. Moreover, we emphasize that we only provide the results and definitions for the doubly reflected case, because the corresponding ones for the upper reflected case can be deduced easily. In the following, we fix a probability measure $\mathbb{P}$

4.6.1 Definitions and first properties

Let us be given the following objects

- A function $g_s(\omega, y, z)$, $\mathbb{F}$-progressively measurable for fixed y and z, uniformly Lipschitz in (y,z) and such that $\mathbb{E}^{\mathbb{P}}\left[\int_0^T |g_s(0,0)|^2\, ds\right] < +\infty$.
- A terminal condition ξ which is $\mathcal{F}_T$-measurable and in $L^2(\mathbb{P})$.
- Càdlàg processes V, S, L in $\mathbb{I}^2(\mathbb{P})$ such that S and L satisfy Assumptions **(E)** and **(F)** (in particular S is a semimartingale with the decomposition (4.2.3)), and with $\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T} |V_t|^2\right] < +\infty$.

We study the problem of finding $(y, z, k^+, k^-) \in \mathbb{D}^2(\mathbb{P}) \times \mathbb{H}^2(\mathbb{P}) \times \mathbb{I}^2(\mathbb{P}) \times \mathbb{I}^2(\mathbb{P})$ such that

$$\begin{aligned}
&y_t = \xi + \int_t^T g_s(y_s, z_s)ds - \int_t^T z_s dW_s + k_T^- - k_t^- - k_T^+ + k_t^+ + V_T - V_t,\ \mathbb{P}-a.s.\\
&L_t \leq y_t \leq S_t,\ \mathbb{P}-a.s.\\
&\int_0^T (S_{s^-} - y_{s^-})\, dk_s^+ = \int_0^T (y_{s^-} - L_{s^-})\, dk_s^- = 0,\ \mathbb{P}-a.s.
\end{aligned}\tag{4.6.36}$$

We first have a result of existence and uniqueness.

Proposition 4.6.1. *Under the above hypotheses, there is a unique solution $(y, z, k^+, k^-) \in \mathbb{D}^2(\mathbb{P}) \times \mathbb{H}^2(\mathbb{P}) \times \mathbb{I}^2(\mathbb{P}) \times \mathbb{I}^2(\mathbb{P})$ to the doubly reflected BSDE (4.6.36).*

Proof. Consider the following penalized RBSDE with lower obstacle L, whose existence and uniqueness are ensured by the results of Lepeltier and Xu [60]

$$y_t^n = \xi + \int_t^T g_s(y_s^n, z_s^n)ds - \int_t^T z_s^n dW_s + k_T^{n,-} - k_t^{n,-} - k_T^{n,+} + k_t^{n,+} + V_T - V_t,$$

where $k_t^{n,+} := n\int_0^t (S_s - y_s^n)^- ds$. Then, define $\widetilde{y}_t^n := y_t^n + V_t$, $\widetilde{\xi} := \xi + V_T$, $\widetilde{z}_t^n := z_t^n$, $\widetilde{k}_t^{n,\pm} := k_t^{n,\pm}$, $\widetilde{g}_t(y,z) := g_t(y - V, z)$ and $\widetilde{L}_t := L_t + V_t$. Then

$$\widetilde{y}_t^n = \widetilde{\xi} + \int_t^T \widetilde{g}_s(\widetilde{y}_s^n, \widetilde{z}_s^n)ds - \int_t^T \widetilde{z}_s^n dW_s + \widetilde{k}_T^{n,-} - \widetilde{k}_t^{n,-} - \widetilde{k}_T^{n,+} + \widetilde{k}_t^{n,+}.$$

Since we know by Lepeltier and Xu [61], that the above penalization procedure converges to a solution of the corresponding RBSDE, existence and uniqueness are then simple generalization. □

We also have a comparison theorem in this context

Proposition 4.6.2. *Let ξ_1 and $\xi_2 \in L^2(\mathbb{P})$, V^i, $i = 1, 2$ be two adapted, càdlàg processes and $g_s^i(\omega, y, z)$ two functions, which verify the above assumptions. Let $(y^i, z^i, k^{i,+}, k^{i,-}) \in \mathbb{D}^2(\mathbb{P}) \times \mathbb{H}^2(\mathbb{P}) \times \mathbb{I}^2(\mathbb{P}) \times \mathbb{I}^2(\mathbb{P})$, $i = 1, 2$ be the solutions of the following DRBSDEs with upper obstacle S^i and lower obstacle L^i*

$$y_t^i = \xi^i + \int_t^T g_s^i(y_s^i, z_s^i)ds - \int_t^T z_s^i dW_s + k_T^{i,-} - k_t^{i,-} - k_T^{i,+} + k_t^{i,+} + V_T^i - V_t^i, \ \mathbb{P} - a.s., \ i = 1, 2,$$

respectively. If it holds $\mathbb{P} - a.s.$ that $\xi_1 \geq \xi_2$, $V^1 - V^2$ is non-decreasing, $S^1 \leq S^2$, $L^1 \geq L^2$ and $g_s^1(y_s^1, z_s^1) \geq g_s^2(y_s^1, z_s^1)$, then we have for all $t \in [0, T]$

$$Y_t^1 \geq Y_t^2, \ \mathbb{P} - a.s.$$

Besides, if $S^1 = S^2$ (resp. $L^1 = L^2$), then we also have $dk^{1,+} \geq dk^{2,+}$ (resp. $dk^{1,-} \leq dk^{2,-}$).

Proof. The first part is classical, whereas the second one comes from the fact that the penalization procedure converges in this framework, as seen previously. Indeed, with the notations of the proof of Proposition 4.6.1, we have in the sense of weak limits

$$k_t^{i,+} = \lim_{n \to +\infty} n \int_0^t \left(S_s - y_s^{n,i}\right)^- ds.$$

Moreover, using the classical comparison Theorem for RBSDEs with the same lower obstacle, we know that $y^{n,1} \geq y^{n,2}$ and $dk_t^{n,1,-} \leq dk_t^{n,2,-}$. This implies that $dk_t^{n,1,+} \geq dk_t^{n,2,+}$. Passing to the limit yields the result. □

Of course, all the above still holds if τ is replaced by some bounded stopping time τ. Following Peng's original ideas, we now define a notion of doubly reflected g-(super)solutions.

Definition 4.6.3. *If y is a solution of a DRBSDE of the form (4.6.36), then we call y a doubly reflected g-supersolution on $[0, \tau]$. If $V = 0$ on $[0, \tau]$, then we call y a doubly reflected g-solution.*

We have the following proposition concerning the uniqueness of a decomposition of the form (4.6.36). Notice that unlike in the lower reflected case considered in [66], the processes V, k^+ and k^- are not necessarily unique.

Proposition 4.6.4. *Given y a g-supersolution on $[0, \tau]$, there is a unique $z \in \mathbb{H}^2(\mathbb{P})$ and a unique couple $(k^+, k^-, V) \in (\mathbb{I}^2(\mathbb{P}))^3$ (in the sense that $V - k^+ + k^-$ is unique), such that (y, z, k^+, k^-, V) satisfy (4.6.36).*

Proof. If both (y, z, k^+, k^-, V) and $(y, z^1, k^{+,1}, k^{-,1}, V^1)$ satisfy (4.6.36), then applying Itô's formula to $(y_t - y_t)^2$ gives immediately that $z = z^1$ and thus $V - k^+ + k^- = V^1 - k^{+,1} + k^{-,1}$, $\mathbb{P} - a.s.$ □

Remark 4.6.5. *We emphasize once more that the situation here is fundamentally different from [66], where reflected g-supersolution were defined for lower reflected BSDEs. In our case, instead of having to deal with the sum of two non-decreasing processes, we actually have to add another non-increasing process. This will raise some difficulties later on, notably when we will prove a non-linear Doob-Meyer decomposition.*

4.6.2 Doob-Meyer decomposition

We now introduce the notion of doubly reflected g-(super)martingales.

Definition 4.6.6. (i) *A doubly reflected g-martingale on $[0,T]$ is a doubly reflected g-solution on $[0,T]$.*

(ii) *A process (Y_t) such that $Y_t \leq S_t$ is a doubly reflected g-supermartingale in the strong (resp. weak) sense if for all stopping time $\tau \leq T$ (resp. all $t \leq T$), we have $\mathbb{E}^{\mathbb{P}}[|Y_\tau|^2] < +\infty$ (resp. $\mathbb{E}^{\mathbb{P}}[|Y_t|^2] < +\infty$) and if the doubly reflected g-solution (y_s) on $[0,\tau]$ (resp. $[0,t]$) with terminal condition Y_τ (resp. Y_t) verifies $y_\sigma \leq Y_\sigma$ for every stopping time $\sigma \leq \tau$ (resp. $y_s \leq Y_s$ for every $s \leq t$).*

Remark 4.6.7. *The above Definition differs once more from the one given in [66]. Indeed, when defining reflected g-supermartingale with a lower obstacle, there is no need to precise that Y is above the barrier L, since it is implied by definition (since y is already above the barrier). However, with an upper obstacle, this is not the case and this needs to be a part of the definition.*

As usual, under mild conditions, a doubly reflected g-supermartingale in the weak sense corresponds to a doubly reflected g-supermartingale in the strong sense. Besides, thanks to the comparison Theorem, it is clear that a doubly reflected g-supersolution on $[0,T]$ is also a doubly reflected g-supermartingale in the weak and strong sense on $[0,T]$. The following Theorem addresses the converse property, which gives us a non-linear Doob-Meyer decomposition.

Theorem 4.6.8. *Let (Y_t) be a right-continuous doubly reflected g-supermartingale on $[0,T]$ in the strong sense with*

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T} |Y_t|^2\right] < +\infty.$$

Then (Y_t) is a doubly reflected g-supersolution on $[0,T]$, that is to say that there exists a quadruple $(z,k^+,k^-,V) \in \mathbb{H}^2(\mathbb{P}) \times \mathbb{I}^2(\mathbb{P}) \times \mathbb{I}^2(\mathbb{P}) \times \mathbb{I}^2(\mathbb{P})$ such that

$$\begin{cases} Y_t = Y_T + \displaystyle\int_t^T g_s(Y_s,z_s)ds + V_T - V_t + k_T^- - k_t^- - k_T^+ + k_t^+ - \int_t^T z_s dW_s \\ L_t \leq Y_t \leq S_t, \ \mathbb{P}-a.s. \\ \displaystyle\int_0^T (S_{s^-} - Y_{s^-})\, dk_s^+ = \int_0^T (Y_{s^-} - L_{s^-})\, dk_s^- = 0. \end{cases} \tag{4.6.37}$$

We then have the following easy generalization of Theorem 3.1 of [79] which will be crucial for our proof.

Theorem 4.6.9. *Consider a sequence of doubly reflected g-supersolutions*

$$\begin{cases} \tilde{y}_t^n = \xi + \displaystyle\int_t^T g_s(\tilde{y}_s^n, \tilde{z}_s^n)ds + \tilde{V}_T^n - \tilde{V}_t^n + \tilde{k}_T^{n,-} - \tilde{k}_t^{n,-} - \tilde{k}_T^{n,+} + \tilde{k}_t^{n,+} - \int_t^T \tilde{z}_s^n dW_s \\ L_t \leq \tilde{y}_t^n \leq S_t \\ \displaystyle\int_0^T (S_{s^-} - \tilde{y}_{s^-}^n)\, d\tilde{k}_s^{n,+} = \int_0^T (\tilde{y}_{s^-}^n - L_{s^-})\, d\tilde{k}_s^{n,-} = 0, \end{cases} \tag{4.6.38}$$

where the $\tilde{V}^n$ are in addition supposed to be continuous. Assume furthermore that

- $(\tilde{y}^n)_{n\geq 0}$ *increasingly converges to* $\tilde{y}$ *with* $\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T} |\tilde{y}_t|^2\right] < +\infty$.
- $d\tilde{k}_t^{n,+} \leq d\tilde{k}_t^{p,+}$ *for* $n \leq p$ *and* $\tilde{k}^{n,+}$ *converges to* $\tilde{k}^+$ *with* $\mathbb{E}^{\mathbb{P}}\left[\left(\tilde{k}_T^+\right)^2\right] < +\infty$.
- $d\tilde{k}_t^{n,-} \geq d\tilde{k}_t^{p,-}$ *for* $n \leq p$ *and* $\tilde{k}^{n,-}$ *converges to some* $\tilde{k}^-$.
- $(\tilde{z}^n)_{n\geq 0}$ *weakly converges in* $\mathbb{H}^2(\mathbb{P})$ *(along a subsequence if necessary) to* $\tilde{z}$.

Then $\tilde{y}$ *is a doubly reflected* g*-supersolution, that is to say that there exists* $\tilde{V} \in \mathbb{I}^2(\mathbb{P})$ *such that*

$$\begin{cases} \tilde{y}_t = \xi + \int_t^T g_s(\tilde{y}_s, \tilde{z}_s)ds + \tilde{V}_T - \tilde{V}_t + k_T^- - k_t^- - \tilde{k}_T^+ + \tilde{k}_t^+ - \int_t^T \tilde{z}_s dW_s,\ \mathbb{P}-a.s. \\ \tilde{y}_t \leq S_t,\ \mathbb{P}-a.s. \\ \int_0^T (S_{s^-} - \tilde{y}_{s^-})\, d\tilde{k}_s = 0,\ \mathbb{P}-a.s.,\ \forall t \in [0,T], \end{cases}$$

Besides, $\tilde{z}$ *is strong limit of* $\tilde{z}^n$ *in* $\mathbb{H}^p(\mathbb{P})$ *for* $p < 2$ *and* $\tilde{V}_t$ *is the weak limit of* $\tilde{V}_t^n$ *in* $L^2(\mathbb{P})$.

Proof. All the convergences are proved exactly as in Theorem 3.1 of [79], using the fact that the sequence of added increasing process $\tilde{k}^{n,-}$ is decreasing. Moreover, since the sequence y^n is increasing it is clear that we have

$$L_t \leq \tilde{y}_t \leq S_t,\ t \in [0,T],\ \mathbb{P}-a.s.$$

We now want to show that we also recover the Skorohod conditions. The proofs being similar, we will only show one of them. We have

$$\begin{aligned} 0 \leq \int_0^T (S_{s^-} - \tilde{y}_{s^-})\, d\tilde{k}_s &= \int_0^T (S_{s^-} - \tilde{y}_{s^-}^n)\, d\tilde{k}_s + \int_0^T (\tilde{y}_{s^-}^n - \tilde{y}_{s^-})\, d\tilde{k}_s \\ &= \int_0^T (S_{s^-} - \tilde{y}_{s^-}^n)\, d\left(\tilde{k}_s - \tilde{k}_s^n\right) + \int_0^T (\tilde{y}_{s^-}^n - \tilde{y}_{s^-})\, d\tilde{k}_s \\ &\leq \int_0^T \left(S_{s^-} - \tilde{y}_{s^-}^0\right) d\left(\tilde{k}_s - \tilde{k}_s^n\right) + \int_0^T (\tilde{y}_{s^-}^n - \tilde{y}_{s^-})\, d\tilde{k}_s. \end{aligned}$$

By the convergences assumed on $\tilde{y}^n$ and $\tilde{k}^n$, the right-hand side above clearly goes to 0 as n goes to $+\infty$, which gives us the desired result. □

Let now Y be a given doubly reflected g-supermartingale. We follow again [77] and we will apply the above Theorem to the following sequence of DRBSDEs

$$\begin{cases} y_t^n = Y_T + \int_t^T \left(g_s(y_s^n, z_s^n) + n(y_s^n - Y_s)^-\right) ds + k_T^{n,-} - k_t^{n,-} - k_T^{n,+} + k_t^{n,+} - \int_t^T z_s^n dW_s, \\ L_t \leq y_t^n \leq S_t, \\ \int_0^T (S_{s^-} - y_{s^-}^n)\, dk_s^{n,+} = \int_0^T (y_{s^-}^n - L_{s^-})\, dk_s^{n,-} = 0. \end{cases} \tag{4.6.39}$$

Our first result is that

Lemma 4.6.10. *For all* n*, we have* $Y_t \geq y_t^n$.

Proof. The proof is exactly the same as the proof of Lemma 3.4 in [77], so we omit it. □

We will now prove some estimates which will allow us to apply Theorem 4.6.9.

Lemma 4.6.11. *There exists a constant $C>0$ independent of n such that the processes defined in (4.6.39) verify*

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}|y_t^n|^2+\int_0^T|z_s^n|^2\,ds+(V_T^n)^2+(k_T^{n,+})^2+(k_T^{n,-})^2\right]\leq C.$$

Proof. First of all, let us define $(\bar{y},\bar{z},\bar{k}^+,\bar{k}^-)$ the unique solution of the DRBSDE with terminal condition Y_T, generator g, upper obstacle S and lower obstacle L (once again, existence and uniqueness are ensured by the results of [26] or [61]). By the comparison Theorem 4.6.2, it is clear that we have for all $n\geq 0$

$$y_t^n\geq \bar{y}_t,\ t\in[0,T],\ \mathbb{P}-a.s.$$

Consider now $(\tilde{y},\tilde{z},\tilde{k}^+,\tilde{k}^-)$ the unique solution of the doubly reflected BSDE with terminal condition Y_T, generator g, upper obstacle S and lower obstacle Y, that is to say

$$\begin{cases}\tilde{y}_t=Y_T+\displaystyle\int_t^T g_s(\tilde{y}_s,\tilde{z}_s)ds+\tilde{k}_T^--\tilde{k}_t^--\tilde{k}_T^++\tilde{k}_t^+-\int_t^T\tilde{z}_sdW_s,\\ Y_t\leq\tilde{y}_t\leq S_t,\\ \displaystyle\int_0^T\left(S_{s^-}-\tilde{y}_{s^-}\right)d\tilde{k}_s^+=\int_0^T\left(\tilde{y}_{s^-}-Y_{s^-}\right)d\tilde{k}_s^-=0.\end{cases}\tag{4.6.40}$$

Notice that since the upper obstacle S is a semimartingale satisfying

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}\left((S_t)^-\right)^2\right]<+\infty,$$

we know from the results of Crépey and Matoussi [26] (see Theorem 3.2 and Proposition 5.2) that the above doubly reflected BSDE has indeed a unique solution and that we have for some constant $C>0$

$$\mathbb{E}^{\mathbb{P}}\left[\left(\tilde{k}_T^+\right)^2\right]\leq C.$$

Moreover, it is clear that since $\tilde{y}_s\geq Y_s$, we also have

$$\tilde{y}_t=Y_T+\int_t^T g_s(\tilde{y}_s,\tilde{z}_s)ds+n\int_t^T(\tilde{y}_s-Y_s)^-ds+\tilde{k}_T^--\tilde{k}_t^--\tilde{k}_T^++\tilde{k}_t^+-\int_t^T\tilde{z}_sdW_s.\tag{4.6.41}$$

Notice also that since Y is a doubly reflected g-supermartingale, we have $y\geq L$. Thus we can use now the comparison Theorem of Proposition 4.6.2 for doubly reflected BSDEs with the same upper obstacles. We deduce that

$$y_t^n\leq\tilde{y}_t,\ \text{and}\ dk_t^{n,+}\leq d\tilde{k}_t^+.$$

Hence, this implies immediately that for some constant C independent of n

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}|y_t^n|^2+\left(k_T^{n,+}\right)^2\right]\leq\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}|\bar{y}_t|^2+\sup_{0\leq t\leq T}|\tilde{y}_t|^2+\left(\tilde{k}_T^+\right)^2\right]\leq C.\tag{4.6.42}$$

Define then $V_t^n := n\int_0^t (y_s^n - Y_s)^- \, ds$. We have

$$\begin{aligned} V_T^n + k_T^{n,-} &= y_0^n - y_T^n - \int_0^T g_s(y_s^n, z_s^n)ds + k_T^{n,+} + \int_0^T z_s^n dW_s \\ &\leq C\left(\sup_{0\leq t\leq T} |y_t^n| + \int_0^T |z_s^n|\, ds + \int_0^T |g_s(0,0)|\, ds + k_T^n + \left|\int_0^T z_s^n dW_s\right|\right). \end{aligned} \quad (4.6.43)$$

Using (4.6.42) and BDG inequality, we obtain from (4.6.43)

$$\mathbb{E}^{\mathbb{P}}\left[(V_T^n)^2 + (k_T^{n,-})^2\right] \leq C_0\left(1 + \mathbb{E}^{\mathbb{P}}\left[\int_0^T |g_s(0,0)|^2\, ds + \int_0^T |z_s^n|^2\, ds\right]\right). \quad (4.6.44)$$

Then, using Itô's formula, we obtain classically for all $\varepsilon > 0$

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}\left[\int_0^T |z_s^n|^2\, ds\right] &\leq \mathbb{E}^{\mathbb{P}}\left[(y_T^n)^2 + 2\int_0^T y_s^n g_s(y_s^n, z_s^n)ds + 2\int_0^T y_{s^-}^n d(V_s^n - k_s^{n,+} + k_s^{n,-})\right] \\ &\leq \mathbb{E}^{\mathbb{P}}\left[C(1 + \sup_{0\leq t\leq T} |y_t^n|^2) + \int_0^T \frac{|z_s^n|^2}{2} ds + \varepsilon(|V_T^n|^2 + |k_T^{n,+}|^2 + |k_T^{n,-}|^2)\right]. \end{aligned}$$

Then, from (4.6.44), we obtain by choosing $\varepsilon = \frac{1}{4C_0}$ that

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T |z_s^n|^2\, ds\right] \leq C.$$

Reporting this in (4.6.44) ends the proof. □

Finally, we can now prove Theorem 4.6.8

Proof. [Proof of Theorem 4.6.8] We first notice that since $Y_t \geq y_t^n$ for all n, by the comparison Theorem for DRBSDEs, we have

$$y_t^n \leq y_t^{n+1},\ dk_t^{n,-} \geq dk_t^{n+1,-} \text{ and } dk_t^{n,+} \leq dk_t^{n+1,+}.$$

By the a priori estimates of Lemma 4.6.11, they therefore converge to some processes y, k^+ and k^-. Moreover, since z^n is bounded uniformly in n in the Banach $\mathbb{H}^2(\mathbb{P})$, there exists a weakly convergent subsequence, and the same holds for $g_t(y_t^n, z_t^n)$. Hence, all the conditions of Theorem 4.6.9 are satisfied and y is a doubly reflected g-supersolution on $[0,T]$ of the form

$$y_t = Y_T + \int_t^T g_s(y_s, z_s)ds + V_T - V_t - k_T^+ + k_t^+ + k_T^- - k_t^- - \int_t^T z_s dW_s,$$

where V_t is the weak limit of $V_t^n := n\int_0^t (y_s^n - Y_s)^- ds$. From Lemma 4.6.11, we have

$$\mathbb{E}^{\mathbb{P}}[(V_T^n)^2] = n^2\mathbb{E}^{\mathbb{P}}\left[\int_0^T \left|(y_s^n - Y_s)^-\right|^2 ds\right] \leq C.$$

It then follows that $Y_t = y_t$ (since we already had $Y_t \geq y_t^n$ for all n), which ends the proof. □

4.6.3 Time regularity of doubly reflected g-supermartingales

In this section we prove a downcrossing inequality for doubly reflected g-supermartingales in the spirit of the one proved in [23]. We use the same notations as in the classical theory of g-martingales (see [23] and [77] for instance).

Theorem 4.6.12. *Assume that $g(0,0) = 0$. Let (Y_t) be a positive reflected g-supermartingale in the weak sense and let $0 = t_0 < t_1 < ... < t_i = T$ be a subdivision of $[0,T]$. Let $0 \leq a < b$, then there exists $C > 0$ such that $D_a^b[Y,n]$, the number of downcrossings of $[a,b]$ by $\{Y_{t_j}\}$, verifies*

$$\mathcal{E}^{-\mu}[D_a^b[Y,n]] \leq \frac{C}{b-a}\mathcal{E}^{\mu}[Y_0 \wedge b],$$

where μ is the Lipschitz constant of g.

Proof. Consider for all $i = 0..n$ the following DRBSDEs with upper obstacle S and lower obstacle L on $[0,t_i]$

$$\begin{cases} y_t^i = Y_{t_i} - \displaystyle\int_t^T (\mu\left|y_s^i\right| + \mu\left|z_s^i\right|)ds + k_{t_i}^{i,-} - k_t^{i,-} - k_{t_i}^{i,+} + k_t^{i,+} - \int_t^{t_i} z_s^i dW_s \ \mathbb{P}-a.s. \\ L_t \leq y_t^i \leq S_t, \ \mathbb{P}-a.s. \\ \displaystyle\int_0^{t_i} \left(y_{s^-}^i - L_{s^-}\right) dk_s^{i,-} = \int_0^{t_i} \left(S_{s^-} - y_{s^-}^i\right) dk_s^{i,+} = 0, \ \mathbb{P}-a.s. \end{cases}$$

By the comparison theorem of Proposition 4.6.2, we know that we have for all i, $y_t^i \geq \tilde{y}_t^i$ for $t \in [0,t_i]$, where $(\tilde{y}^i, \tilde{z}^i, \tilde{k}^i)$ is the unique solution of the RBSDE on $[0,t_i]$ with the same generator and terminal condition as above and upper obstacle S, that is to say

$$\begin{cases} \tilde{y}_t^i = Y_{t_i} - \int_t^T (\mu\left|\tilde{y}_s^i\right| + \mu\left|\tilde{z}_s^i\right|)ds - \tilde{k}_{t_i}^i + \tilde{k}_t^i - \int_t^{t_i} \tilde{z}_s^i dW_s \ \mathbb{P}-a.s. \\ \tilde{y}_t^i \leq S_t, \ \mathbb{P}-a.s. \\ \int_0^{t_i} \left(S_{s^-} - \tilde{y}_{s^-}^i\right) d\tilde{k}_s^i = 0, \ \mathbb{P}-a.s. \end{cases}$$

We define $a_s^i := -\mu \ \mathrm{sgn}(z_s^i) 1_{t_{j-1} < s \leq t_j}$ and $a_s := \sum_{i=0}^n a_s^i$. Let $\mathbb{Q}^a$ be the probability measure defined by

$$\frac{d\mathbb{Q}^a}{d\mathbb{P}} = \mathcal{E}\left(\int_0^T a_s dW_s\right).$$

We then have easily that $y_t^i \geq 0$ since $Y_{t_i} \geq 0$. We next define $\hat{y}^i := \tilde{y}^i - k^i$, $\hat{Y}^i := Y - k^i$. Then, $(\hat{y}^i, z^i)$ solves the following BSDE on $[0,t_i]$

$$\hat{y}_t^i = \hat{Y}_{t_i}^i - \int_t^{t_i} \mu\left(\hat{y}_s^i + k_s^i\right) + \mu\left|z_s^i\right| ds - \int_t^{t_i} z_s^i dW_s.$$

It is then easy to solve this BSDE to obtain

$$\hat{y}_t^i = \mathbb{E}_t^{\mathbb{Q}^a}\left[e^{-\mu(t_i - t)}\hat{Y}_{t_i}^i - \mu\int_t^{t_i} e^{-\mu(s-t)} k_s^i ds\right].$$

Define now the following càdlàg process $k_t := \sum_{i=i}^n k_t^i 1_{t_{i-1}\leq t<t_i}$ and $\hat{Y} := Y - k$. We clearly have for $t = t_{i-1}$

$$\hat{y}^i_{t_{i-1}} = \mathbb{E}^{\mathbb{Q}^a}_{t_{i-1}}\left[e^{-\mu(t_i-t_{i-1})}\hat{Y}_{t_i} - \mu\int_{t_{i-1}}^{t_i} e^{-\mu(s-t_{i-1})}k_s ds\right].$$

Now, since Y is a doubly reflected g-supermartingale (and thus also a doubly reflected $g^{-\mu}$-supermartingale where $g_s^{-\mu}(y,z) := -\mu(|y|+|z|)$ by a simple application of the comparison theorem), we have

$$\hat{y}^i \leq y^i - k^i \leq \hat{Y}.$$

Hence, we have obtained

$$\mathbb{E}^{\mathbb{Q}^a}_{t_{i-1}}\left[e^{-\mu(t_i-t_{i-1})}\hat{Y}_{t_i} - \mu\int_{t_{i-1}}^{t_i} e^{-\mu(s-t_{i-1})}k_s ds\right] \leq \hat{Y}_{t_{i-1}}.$$

This actually implies that the process $X := (X_{t_i})_{0\leq i\leq n}$ where

$$X_{t_i} := e^{-\mu t_i}\hat{Y}_{t_i} - \mu\int_0^{t_i} e^{-\mu s}k_s ds,$$

is a $\mathbb{Q}^a$-supermartingale. Then we can finish the proof exactly as in the proof of Theorem 6 in [23]. $\square$

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